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HOW FAR CAN ONE GET WITH A LINEAR FIELD THEORY OF GRAVITATION IN FLAT SPACE-TIME?[1]

By Hermann Weyl.

Introduction and Summary. G. D. Birkhoff's atempt to establish a linear field theory of gravitation within the frame of special relativity[2] makes it desirable to probe the potentialities and limitations of such a theory in more general terms. In thus continuing a discussion begun in another place[3] I find that the differential operators at one's disposal form a 5 dimensional linear manifold. But the requirement that the field equations imply the law of conservation of energy and momentum in the simple form $\partial T_i^k/\partial x_k = 0$ limit these ∞^5 possibilities to ∞^2, which, however, reduce to two cases, a regular one (L) and a singular one (L'). The regular case (L) is nothing but Einstein's theory of weak fields. Resembling very closely Maxwell's theory of the electromagnetic field, it satisfies a principle of gauge involving 4 arbitrary functions, and although its gravitational field exerts no force on matter, it is well suited to illustrate the role of energy and momentum, charge and mass in the interplay bwtween matter and field. It might also help, though this is much more problematic, in pointing the way to a more satisfactory unification of gravitation and electricity than we at present possess. Birkhoff follows the opposite way: by avoiding rather than adopting the ∞^2 special operators mentioned above, his "'dualistic"' theory B) destroys the bond between mechanical and field equations, which is such a decisive feature in Einstein's theory.

1. Maxwell's theory of the electromagnetic field and the monistic linear theory of gravitation (L). Gauge invariance. Within the frame of special relativity and its metric ground form

$$ds^2 = \delta_{ik} dx_i dx_k = dx_o^2 - \left(dx_1^2 + dx_2^2 + dx_3^2\right)$$

an electromagnetic field is described by a skew tensor

$$f_{ik} = \partial\phi_k/\partial x_i - \partial\phi_i/\partial x_k$$

derived from a vector potential ϕ_i and satisfies Maxwell's equations

$$\partial f^{ki}/\partial x_k = s^i \text{ or } D_i\phi = \Box\phi_i - \partial\phi'/\partial x_i = s_i \tag{1}$$

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[2]*Proceedings of the National Academy of Sciences*, vol. 29 (1943), p.231.

[3]*Proceedings of the National Academy of Sciences*, vol. 30 (1944), p205.

where s^i is the density-flow of electric charge and

$$\phi' = \partial\phi^i/\partial x_i, \quad \Box\phi = \delta^{pq}\left(\partial^2\phi/\partial x_p \partial x_q\right).$$

The equations do not change if one substitutes

$$\phi_i^* = \phi_i - \partial\lambda/\partial x_i \quad for \quad \phi_i, \tag{2}$$

λ being an arbitrary function of the coordinates ("'*gauge invariance*"'), and they imply the differential conservation law of electric charge:

$$\partial s^i/\partial x_i = 0. \tag{3}$$

As is easily verified, there are only two ways in which one may form a vector field by linear combination of the second derivatives of a given vector field ϕ_i, namely

$$\Box\phi_i \quad and \quad \partial\phi'/\partial x_i \qquad (\phi' = \partial\phi^p/\partial x_p).$$

Herein lies a sort of mathematical justification for Maxwell's equations.

Taking from Einstein's theory of gravitation the hint that gravitation is represented by a symmetric tensor potential h_{ik}, but trying to emulate the linear character of Maxwell's theory of the electromagnetic field, one could ask oneself what symmetric tensors $\overline{D}_{ik}h$ can be constructed by linear combination from the second derivatives of h_{ik}. The answer is that there are 5 such expressions, namely

$$\Box h_{ik}, \quad \partial h_i'/\partial x_k' + \partial h_k'/\partial x_i', \quad h''\delta_{ik}, \quad \partial^2 h/\partial x_i \partial x_k, \quad \Box h \cdot \delta_{ik} \tag{4}$$

where

$$h = h_p^p, \quad h_i' = \partial h_i^p/\partial x_p, \quad h'' = \partial^2 h^{pq}/\partial x_p \partial x_q.$$

With any linear combination $\overline{D}_{ik}h$ of these 5 expressions one could set up the field equations of gravitation

$$\overline{D}_{ik}h = T_{ik} \tag{5}$$

the right member of which is the energy-momentum tensor T_{ik}. In analogy to the situation encountered in Maxwell's theory one may ask further for which linear combinations $\overline{D}_{ik}$ the identity

$$(\partial/\partial x_k)\left(\overline{D}_i^k h\right) = 0$$

will hold, and one finds that this is the case if, and only if $\overline{D}_{ik}$ is of the form

$$\alpha\left\{\Box h_{ik} - (\partial h_i'/\partial x_k + \partial h_k'/\partial x_i) + h''\delta_{ik}\right\} + \beta\left\{\partial^2 h/\partial x_i \partial x_k - \Box h \cdot \delta_{ik}\right\}, \tag{6}$$

α and β being arbitrary constants. In this case the field equations (5) entail the differential conservation law of energy and momentum

$$\partial T_i^k/\partial x_k = 0. \tag{7}$$

With the constants a, b $(a \neq 0, a \neq 4b)$ we can make the substitution

$$h_{ik} \to a \cdot h_{ik} - b \cdot h\delta_{ik}$$

and thereby reduce α, β to the values $1, 1$, provided $\alpha \neq 0, \alpha \neq 2\beta$. Hence, disregarding these singular values, we may assume as our field equations

$$\overline{D}_{ik}h \equiv \{\Box h_{ik} - (\partial h'_i/\partial x_k + \partial h'_k/\partial x_i) + h''\delta_{ik}\} + \left\{\partial^2 h/\partial x_i \partial x_k - \Box h \cdot \delta_{ik}\right\} = T_{ik}. \qquad (\overline{5})$$

$\Box h_{ik}$ remains unchanged if h_{ik} is replaced by

$$h^*_{ik} = h_{ik} + (\partial\xi_i/\partial x_k + \partial\xi_k/\partial x_i) \qquad (8)$$

where ξ_i is an arbitrary vector field. Hence we have the same type of correlation between gauge invariance and conservation law for the gravitational field as for the electromagnetic field, and it is reasonable to consider as physically equivalent any two tensor fields h, h^* which are related by (8).

The linear theory of gravitation (L) in a flat world at which one thus arrives with a certain mathematical necessity is nothing else but Einstein's theory for weak fields. Indeed, on replacing Einstein's g_{ik} by $\delta_{ik} + 2\kappa \cdot h_{ik}$ and then neglecting higher powers of the gravitational constant κ, one obtains $(\overline{5})$, and the property of gauge invariance (8) reflects the invariance of Einstein's equations with respect to arbitrary coordinate transformations.[4]

By proper normalization of the arbitrary function λ in (2) one may impose the condition $\phi' = 0$ upon the ϕ_i, thus giving Maxwell's equations a form often used by H. A. Lorentz:

$$\Box\phi_i = s_i, \qquad \partial\phi^i/\partial x_i = 0. \qquad (9)$$

In the same manner one can choose the ξ_i in (8) so that $\gamma_{ik} = h_{ik} - \frac{1}{2}h \cdot \delta_{ik}$ satisfies the equations

$$\partial\gamma^k_i/\partial x_k = 0 \quad \textit{and} \qquad (10)$$

$$\Box\gamma_{ik} = T_{ik}. \qquad (11)$$

In one important respect gauge invariance works differently for electromagnetic and gravitational fields: If one splits the tensor of derivatives $\phi_{k,i} = \partial\phi_k/\partial x_i$ into a skew and symmetric part,

$$\phi_{k,i} = \frac{1}{2}(\phi_{k,i} - \phi_{i,k}) + \frac{1}{2}(\phi_{k,i} + \phi_{i,k}),$$

the first part is not affected by a gauge transformation whereas the second can locally be transformed into zero. In the gravitational case *all* derivatives $\partial h_{ik}/\partial x_p$ can locally be transformed into zero. Hence we may construct, according to

[4]Cf. A. Einstein, *Sitzungsber. Preuss. Ak. Wiss.* (1916), p. 688 (and 1918, p. 154).

Faraday and Maxwell, an energy-momentum tensor L_{ik} of the electromagnetic field,

$$L_i^k = f_{ip}f^{pk} - \frac{1}{2}\delta_i^k\,(ff)\,, \qquad (ff) = \frac{1}{2}f_{pq}f^{qp}, \tag{12}$$

depending quadratically on the gauge invariant field components

$$f_{ik} = \phi_{k,i} - \phi_{i,k},$$

but no tensor G_{ik} depending quadratically on the derivatives $\partial h_{ik}/\partial x_p$ exists, if gauge invariance is required, other than the trivial $G_{ik} \equiv 0$.

2. Particles as centers of force, and the charge vector and energy-momentum tensor of a continuous cloud of substance. Conceiving a resting particle as a center of force, let us determine the *static centrally symmetric solutions* of our homogeneous field equations (1) and $(\overline{5})$ ($s^i = 0, T_{ik} = 0$). One easily verifies that *in the sense of equivalence* the most general such solution is given by the equations

$$\phi_0 = e/4\pi r. \quad \phi_i = 0 \ for\ i \neq 0; \tag{13}$$

$$\gamma_{00} = m/4\pi r, \quad \gamma_{ik} = 0\ for\ (i,k) \neq (0,0), \tag{14}$$

r being the distance from center. As was to be hoped, it involves but two constants, *charge* e and *mass* m. The center itself appears as a singularity in the field. Indeed ϕ_0 and the factor ϕ in $\phi_\alpha = \phi x_\alpha$ $[\alpha = 1,2,3]$ must be functions of r alone, and the relations

$$\Delta\phi_0 = 0, \qquad \partial\phi_\alpha/\partial x_\alpha = 0 \qquad [\alpha = 1,2,3]$$

implied in (9) then yield

$$\phi_0 = a/r, \qquad \phi = b/r^3, \qquad \phi_\alpha = -\left(\partial/\partial x_\alpha\right)(b/r)\,.$$

Substitution of $\phi_\alpha - \partial\lambda/\partial x_\alpha$ for ϕ_α with $\lambda = -b/r$ changes ϕ_α into zero. In the same manner (14) is obtained from the equations (10 and 11).

A continuous cloud of 'charged dust' can be characterized by its velocity field u^i ($u_i u^i = 1$) and the rest densities μ, ρ of mass and charge. It is well known that its equations of motion and the differential conservation laws of mass and charge result if one sets $s^i = \rho u^i$ in Maxwell's equations and lets T_i^k in (7) consist of the Faraday-Maxwell field part (12) and the kinetic part $\mu u_i u^k$:

$$\partial(\rho u^i)/\partial x_i = 0, \qquad \partial(\mu u^i)/\partial x_i = 0;$$

$$\mu du_i/ds = \rho \cdot f_{ip}u^p.$$

Since the motion of the individual dust particle is determined by $dx_i/ds = u^i$ we have written d/ds for $u^k\partial/\partial x_k$. In this manner Faraday explained by his electromagnetic tensions (flow of momentum) the fact that the *active* charge which generates an electric field is at the same time the *passive* charge on which a given field acts. At its present stage our theory (L) accounts for the force which an electromagnetic field exerts upon matter, but the gravitational field remains a powerless shadow. From the standpoint of Einstein's theory this is as it should be, because the gravitational force arises only when one continues the approximation beyond the linear stage. We pointed out above that no remedy for this defect may be found in a gauge invariant gravitational energy-momentum tensor. However, the theory (L) explains why active gravity, represented by the scalar factor μ in the kinetic term $\mu u_i u_k$ as it appears in the right member T_{ik} of the gravitational equations ($\overline{5}$), is at the same time inertial mass: this is simply another expression of the fact that the mechanical equations (7) are a consequence of those field equations.

We have seen that even in empty space the field part of energy and momentum must not be ignored, and thus a particle should be described by the static centrally symmetric solution of the equations

$$D_i\phi = 0, \qquad D_{ik}h - L_{ik} = 0 \tag{15}$$

(of which the second set is no longer strictly linear!). Again we find, after proper gauge normalization,

$$\phi_0 = e/4\pi r, \qquad \phi_i = \phi_2 = \phi_3 = 0, \tag{13}$$

and then

$$\gamma_{00} = m/4\pi r - \frac{1}{4}(e/4\pi r)^2, \qquad \gamma_{0\alpha} = 0,$$

$$\gamma_{\alpha\beta} = -(e/4\pi r)^2 \cdot (x_\alpha x_\beta/4r^2), \qquad [\alpha, \beta\ = 1, 2, 3]. \tag{14e}$$

As before, two characteristic constants e and m appear. *At distances much larger than the 'radius' $e^2/4\pi m$ of the particle the gravitational influence of charge becomes negligible compared with that of mass.*

3. The singular case. In normalizing the operator (6) by $\alpha = \beta = 1$ we had to exclude the cases $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 1/2$. The first is clearly without interest because it deals with a field described by a scalar h rather than a tensor h_{ik}. But the differential operator (6), D'_{ik} corresponding to the values $\alpha = 1, \beta = 1/2$ and the attendant field equations

$$D'_{ik}h = T_{ik} \tag{5'}$$

deserve a moment's attention. $D'_{ik}h$ remains unchanged if h_{ik} is replaced by

$$h^*_{ik} = h_{ik} + \eta\delta_{ik} + (\partial\xi_i/\partial x_k + \partial\xi_k/\partial x_i)$$

where 5 functions η, ξ_i are subject to the one restriction $\partial\xi^i/\partial x_i = 0$. By proper gauge normalization one may reduce the field equations (5') to the form

$$\partial h^k_i/\partial x_k = 0, \tag{10'}$$

$$\Box h_{ik} + \tfrac{1}{2}\left(\partial^2 h/\partial x_i \partial x_k - \Box h \cdot \delta_{ik}\right) = T_{ik}. \qquad (11')$$

The static centrally symmetric solution of the homogeneous equations ($T_{ik} = 0$) is the following counterpart to (14):

$$h_{oo} = 0,\ h_{o\alpha} = 0,\ h_{\alpha\beta} = (m'/4\pi r)(\delta_{\alpha\beta} - x_\alpha x_\beta / r^2) \qquad [\alpha, \beta = 1, 2, 3]$$

The same electric part as in (14e) may be superimposed. It seems remarkable that besides (L) this possibility (L') exists.

4. Derivation of the mechanical laws without hypotheses about the inner structure of particles. In principle the idea of substance had already been overcome by Newton's dynamical interpretation of Nature. His particles are centers of force, the inertial mass is a dynamic coefficient and not, as the scholastic definition pretends, quantity of substance. Boscovitch, Ampère and others took the extreme view that the centers of force are points without extension. Modern atomistic physics has raised the discrete structure of matter above all doubt. Although it does not forbid us to picture the elementary particles as something of continuous extension, one must admit that, so far, speculation about their 'interior' have never borne fruit. Indeed we can explain the laws of reaction of particles with the continuous field without committing ourselves to any hypotheses concerning their inner structure, simply *by describing a particle through the surrounding 'local' field.* I proceed to illustrate this fundamental point first by Maxwell's equations and then by our linear theory (L).

A particle describes a narrow channel in the 4 dimensional world. The only assumption concerning the electromagnetic potential ϕ_i we make is that outside this channel Maxwell's homogeneous equations

$$\partial f_{ki}/\partial x_k = 0 \qquad (16)$$

are satisfied. By arbitrary continuous extension we fill the channel with a *fictitious field* ϕ_i and then *define* s^i by (1). The relation (3) is a consequence of this definition, and (16) asserts that s^i vanishes outside the channel. Let S_t denote the plane $x_o = const. = t$, S_t^* the portion of S_t inside the channel, Ω the surface of the channel and Ω_t the intersection of Ω with S_t (or the boundary of S_t^*). The surface Ω_t surrounds the particle in the 3-space S_t. Integrating (3) over S_t we find

$$de/dt = 0 \qquad for \qquad e = \int\int\int_{S_t^*} s^0 dx_1 dx_2 dx_3;$$

hence e does not vary in time. More generally, it can be stated that the vector field s^i sends the same flow e through any 3 dimensional surface crossing the channel. Application of this fact to two different cross sections S_t confirms the above result; application to two cross section $x_0 = const.$ and $x_0^* = const.$ corrseponding to two different admissible coordinate systems x and x^* (which are linked by a lorentz transformation) proves e to be an *invariant.* Finally we must show that it is independent of the fictitious 'filling'. But according to the definition of s^0,

$$e = \int\int\int_{S_t^*} \left(\partial f^{01}/\partial x_1 + \partial f^{02}/\partial x_2 + \partial f^{03}/\partial x_3\right) dx_1 dx_2 dx_3$$

is the flow of the electric field $(\partial f^{01}, \partial f^{02}, \partial f^{03})$ through Ω_t and hence is completely determined by the *real* field on Ω. For this introduction of the charge e it does not matter whether the particle is an actual singularity of the field or covers a (small) region where the known laws in empty space are suspended (and unknown laws take their place). If the field surrounding the particle is described by (13) the the flow e of the electric field through Ω_t is the constant designated by the same letter in (13). Approximately one can ascribe a world direction u^i to the channel, and it is clear that, if numerous particles of nearly the same velocity u^i, each with its charge e, are encountered in a macroscopic 'volume element' of space, their effect can macroscopically be accounted for by a convective current ρu^i.

Faute de mieux, H. A. Lorentz and H. Poincarè used this expression also for the infinitesimal volume elements of an electron, and the question arose by what cohesive forces the charges of the several parts of an electron are held together against their electrostatic repulsion. Compared with this primitive viewpoint (which was elaborated in considerable detail by M. Abraham) G. Mie's field theory of particles[5], which expressed the current s^i in terms of the same fundamental quantities, namely ϕ_i, as the field itself, signified an enormous progress. But also this theory, in spite of some highly attractive features, the great hopes it once raised and its development by men like D. Hilbert, M. Born and others, has remained in the limbo of speculative physics. The sober non-committal attitude here described was the third stage in the history of our problem. [A fourth has been opened by quantum physics: Following Schrödinger's footsteps, Dirac expressed s^i in terms of the 4 spinor components of the electronic field ψ. This is a simple extension of the scheme of field physics, which in itself is as natural as the appearance of the Maxwellian L_{ik} in the gravitational field equations (15). However, an entirely new feature, statistical interpretation based on quantization of the field laws, 'creates' in quantum physics the discrete particles. The singularities to which this process of quantization gives rise constitute a difficulty at least as serious in quantum as in classical physics.]

Let us return to the classical standpoint and proceed from electricity to gravitation. After bridging the channel by a fictitious field h_{ik} we integrate the identities

$$(\partial/\partial x_k)(D_i^k h) = 0$$

over a cross section S_t^* of the channel, thus obtaining the mechanical equations

$$dJ_i/dt = P_i \tag{16}$$

in which

$$J_i = \int\int\int_{S_t^*} D_i^0 h \cdot dx_1 dx_2 dx_3$$

[5] *Ann. d. Phys.*, vols. 37, 39, 40 (1912/13).

and -P_i is the flow of the vector field $(D_i^1 h, D_i^2 h, D_i^3 h)$ on S_t through Ω_t. By its definition P_i does not depend on the fictitious filling, and from this fact and (16) it follows that the same is true for J_i. Indeed define $J_i^{(1)}$ by a filling 1, $J_i^{(2)}$ by a filling 2, consider two distinct cross sections $S_1, S_2, t = t_1$ and $t = t_2$, and construct a filling 3 that coincides with 1 in the neighborhood of S_1, with 2 in the neighborhood of S_2. Applying (16) to these three fillings and recalling that P_i remains unaffected one finds

$$J_i^{(1)}(t_2) - J_i^{(1)}(t_1) = \int_{t_1}^{t_2} P_i dt, \quad J_i^{(2)}(t_2) - J_i^{(2)}(t_1) = \int_{t_1}^{t_2} P_i dt,$$

$$J_i^{(3)}(t_2) - J_i^{(3)}(t_1) = J_i^{(2)}(t_2) - J_i^{(1)}(t_1) = \int_{t_1}^{t_2} P_i dt;$$

hence

$$J_i^{(1)}(t) = J_i^{(2)}(t) \ for \ t = t_1 \ and \ t_2.$$

When dealing with an *isolated* system we can assume that $D_{ik}h$ vanishes outside the channel; then $P_i = 0$. Let us choose an arbitrary constant contravariant vector l^i and form the vector field $q^k = l^i \cdot D_i^k h$, which satisfies the equation $\partial q^k / \partial x_k = 0$ and under our assumption vanishes outside the channel. The argument previously applied to s^k proves that

$$\int\int\int_{S_t^*} q^0 dx_1 dx_2 dx_3 = l^i J_i$$

is constant in time and an invariant. Hence J_i *are the components of a covariant vector*. In this way we introduce the energy-momentum vector J of an isolated particle and obtain the concervation law

$$J_i = const. \tag{17}$$

For the *static* field (14) one may compute J_i by means of a *static* filling. Then $J_1 = J_2 = J_3 = 0$ and J_0 is the integral of

$$D_0^0 h = -\Delta\gamma_{00} + \partial^2 \gamma^{\alpha\beta} / \partial x_\alpha \partial x_\beta \qquad [\alpha, \beta = 1, 2, 3]$$

over a sphere S_0^* around the center, hence the flow through its surface Ω_0 of the spatial vector

$$-\left\{\partial\gamma_{00}/\partial x_\alpha + \partial\gamma_\alpha^\beta / \partial x_\beta\right\}.$$

But this flow may be computed from the *real* field and thus turns out to be radial and of strength $m/4\pi \cdot 1/r^2$; consequently $J_0 = m$.

Since J_i is a covariant vector, our result $J_0 = m, J_1 = J_2 = J_3 = 0$ carries over from a resting isolated particle to one moving in the direction u^i:

$$J_i = mu^i. \tag{18}$$

For a particle interacting with other particles we can not assume that $D_{ik}h$ vanishes outside the channel, and the conservation law (17) must be replaced by the mechanical equations (16). We might call P external force and J energy-momentum; both, as we have seen, are independent of the filling, but there is no reason why J should be a vector. We get beyond this general scheme by an approximate evaluation of P and J, based on the field equations (15) which hold outside Ω and the character of the local field surrounding the particle. Computation of J_0 for the static centrally symmetric field (14e) by the same method as for the special case $e = 0$ yields

$$J_0 = m - \frac{1}{2} \cdot \left(e^2/4\pi a\right),$$

provided Ω_0 is the sphere of radius a. Notice that $J_0(a)$ tnds to $-\infty$ and not to zero with $a \to 0$. The energy between two spheres of different radii a has the correct value of the electric field energy $(e^2/8\pi)[1/a]$; *nevertheless the total energy* $(a \to 0)$ *is not infinite but* m.

The electric field will be a superposition of the local fields generated by the several particles. In terms of a suitable system of coordinates in which the particle under consideration momentarily (for $t\ = 0$) rests we shall, therefore, have a field $F_{ik} + f_{ik}$ on $\Omega_0 = \Omega_{t=0}$ where

$$(f_{01}, f_{02}, f_{03}) = (e/4\pi r^3)(x_1, x_2, x_3), \qquad f_{12} = f_{23} = f_{31} = 0,$$

while F_{ik} is practically constant, i.e. varies on Ω_0 essentially less than f_{ik} (though it may well be stronger than f_{ik}). A familiar calculation then gives for the flow of

$$-(D_i^1 h, D_i^2 h, D_i^3 h) = -(L_i^1, L_i^2, L_i^3)$$

the value $P_i = eF_{io}$.

Were f_{ik} the total electric field we could assume that the (local) gravitational field surrounding the particle, for $t = 0$ and outside Ω_o, is given by (14e), and we should obtain

$$J_o = m, \qquad J_1 = J_2 = J_3 = 0, \tag{19}$$

provided *the radius* a *of the sphere* Ω_o *is large in comparison with the radius* $e^2/4\pi m$ *of the particle.* We fix Ω_o in this manner: it is at this point that the necessity for keeping away from the particle arises. The equations (14) will still hold with sufficient accuracy on and outside Ω_o if not only e^2/a^4 but also *the energy of the 'outer' field* ΣF_{ik}^2 *on* Ω_o *is small compared to* m/a^3.

Cut the channel by two cross sections $x_o = const.$, $x_o^* = const.$, belonging to two different coordinate systems x, x^* and going through a common point inside the channel. Let l again be an arbitrary constant contravariant vector with the components l^i in the one, l^{*i} in the other coordinate system. The difference of the respective integrals $l^i J_i, l^{*i} J_i^*$ is the flow of

$$(l_i \cdot D_i^1 h, l_i \cdot D_i^2 h, l_i \cdot D_i^3 h) = (l^i L_i^1, l^i L_i^2, l^i L_i^3)$$

through the part of the channel surface Ω between these two cross cuts, and hence, under the above assumptions, of a lower order of magnitude than m.

With this approximatoin J_i is a covariant vector, and thus the formula (18) becomes applicable not only for the cross section $t = 0$ where the particle rests momentarily, but for any cross section $x_o t = const$.

Of course, (16) has to be interpreted in integral fashion,

$$[J_i] = J_i(t_1) - J_i(0) = \int_0^{t_1} P_i dt,$$

and here we may set, with sufficient approximation, $J_i(t) = m(t)u_i(t)$. The equation itself shows that an appreciable change of J_i, one that is comparable with m, can be expected only after a lapse of time t_1 of order $m/e\ |F|$, which is large in comparison with the radius a of Ω_0: Our assumptions imply that J_i or m and u^i change but slowly (*quasi-stationary motion*).

But with these precautions in mind, the differential equation

$$(d/dt)(mu_i) = eF_{i0} \tag{20}$$

may now be claimed as holding for $t = 0$. The component $i = 0$ gives $dm/dt = 0$; hence *the mass m stays constant.* By a known simple technique (20) is changed into its invariant form

$$mdu_i/ds = e \cdot F_{ip}u^p$$

which will hold along the entire channel. The deduction indicates clearly the hypotheses to which the approximate validity of this Lorentz equation of motion of a particle is bound.[6] We now understand why quantities of the type $s_i = \rho u^i, T_i^k = \mu u_i u^k$ can account in a rough manner for the interaction between field and a cloud of charged dust in which near particles have nearly the same velocity.

5. Vague suggestions about a future unification of gravitation and electromagnetism. In spite os such achievements nobody will believe in the sufficiency of the linear theory (L). For, as we have said above, its gravitational field is a shadow without power. the fundamental fact that *passive gravity and inertial mass* always coincide appears to me convincing proof that *general relativity* is the only remedy for the shortcoming. But thereby the gravitational constant κ enters the picture, and one knows that the ratio of the electric and gravitational radii of an electron, (e^2/m): $\kappa m = e^2/\kappa m^2$, is a pure number of the order of magnitude 10^{40}. This circumstance and Mach's old idea that tjhe plane of the Foucault pendulum is carried around by the stars in their daily revolution, point to a construction in which the gravitational force is bound to the totality of masses in the universe. Our present theory, Maxwell + Einstein, with its inorganic juxtaposition of electromagnetism and gravitation, cannot be the last

[6] I have repeated here for the linear theory an argument which I first developed within the frame of general relativity in the 4th and in more detail in the 5th edition of my book 'Raum Zeit Materie'; see the latter edition, Berlin 1923, pp. 277-286. the purely gravitational case was treated with the greatest care in a more recent paper by A. Enistein, L. Infeld and B. Hoffmann, *Annals of Mathematics*, vol. 39 (1938), pp. 65-100.

word. Such juxtaposition may be tolerable for the linear approximation (L) but not in the final generally relativistic theory. Transition from (L) with its flat world to general relativity should raise both, not ony the gravitational, but also the electromagnetic part, above the linear level and, as it changes the gauge transformations of the former into non-linear transformations of coordinates, something similar ought to happen to the gauge transformations of the ϕ_i.

After adding Dirac's 4 spin components to the electronic field ψ to the fundamental field quantities ϕ_i, h_{ik} the electric gauge invariance[7] states that the field equations do not change under the substitution of

$$e^{i\lambda} \cdot \psi, \qquad \phi_k - \frac{h}{e}\frac{\partial \lambda}{\partial x_k} \qquad for \qquad \psi, \phi_k$$

($h =$ Planck's quantum of action): the process of 'covariant derivation' of ψ is defined by $\partial/\partial x_k + (ie/h)\phi_k$. Thus the electromagnetic field ϕ_i appears as a sort of appendage of the ψ-field. It is natural to expect the h_{ik} to be appended in a similar manner to quantities associated with other elementary particles. Thus incompleteness of our present theory on the linear level, a *premature* transition to general relativity, might have their share in blocking the view towards a satisfactory unification. For these reasons a linear theory of gravitation like (L), though necessarily preliminary in character, may still deserve the physicist's attention.

6. A free paraphrase of Birkhoff's recent linear theory of gravitation (B)**.** The linear theory (B), however, is essentially different from (L). It seems to me characteristic for Birkhoff's conception that he uses the kinetic quantities $s^i = \rho u^i, T_i^k = \mu u_i u^k$ not only for a macroscopic description of matter, but a late follower of Lord Kelvin, even for the construction of fluid models of atoms, and that he preserves the duality of field and matter also in the form of mechanical equations which do not follow from the field equations. In contrast to this 'dualistic' scheme Einstein's theory and its linear approximation (L) are 'monistic'.

Since Birkhoff wishes to avoid the fact that mechanical equations such as (7) follow from the field equations, he must choose for the left side $\overline{D}_{ik}h$ of his linear equations (5) any combination of the 5 tensors (4) which is *not* of the special form (6). He picks, somewhat arbitrarily, $\Box h_{ik}$ or rather $\Box h_{ik} - \frac{1}{2}\Box h \cdot \delta_{ik}$; but it seems wiser not to commit oneself too early. He is then at liberty to add to the left member of (7) a term representing the action of the gravitational field on matter. Assuming that force to be quadratic in u^i, as in Einstein's theory, he writes

$$(\partial/\partial x_k)(\mu u_i u^k + L_i^k) + \Gamma_{i,pq} u^p u^q = 0 \tag{21}$$

[7] This principle of gauge invariance is analogous to one by which the author in 1918 made the first attempt at a unification of electromagnetism and gravitation. He has long since realized that it does not connect electricity and gravitation (ϕ_i and g_{ik}), as he then believed, but the electric with the electronic field (ϕ_i with ψ). In this form, in which the exponent of the gauge factor $e^{i\lambda}$ is pure imaginary and not real, it expresses well established atomistic facts, and the connecting coefficient, h/e, is a known atomistic and not an unknown cosmologic constant.

and finds

$$\Gamma_{i,pq} = (\sigma/2)(\partial h_{ip}/\partial x_q + \partial h_{iq}/\partial x_p - 2\partial h_{pq}/\partial x_i) \tag{22}$$

as the mathematically simplest expression by which the differential law of conservation of mass

$$(\partial/\partial x_k)(\mu u^k) = 0$$

is upheld. (In Einstein's theory one has instead $\Gamma_{i,pq} = -\frac{1}{2}\frac{\partial g_{pq}}{\partial x_i}$, provided μ and L_i^k denote the scalar and tensorial *densities*, not scalar and tensor.) But since no theory in which inertia and gravitation are separate entities can explain the universal proportionality of passive gravity and inertial mass, there is no reason why the scalar field σ should be the same as μ (instead one might expect that for a substance *of given chemical constitution* μ and σ are connected by some equation of state $F(\mu, \sigma) = 0$). However, just as Maxwell's L_i^k accounts for the identity of active and passive charge, one can hope in this theory to establish the identity of active and passive gravity by a gravitational energy tensor. For that purpose it is necessary to assume $\sigma u_i u_k$ rather than $\mu u_i u_k + L_{ik}$ as the right member T_{ik} of the field equations (5),

$$\overline{D}_{ik} h = \sigma u_i u_k,$$

and one will try to construct a symmetric tensor G_{ik} which is quadratic in the derivatives $\partial h_{pq}/\partial x_k$ such that the following identity holds:

$$\partial G_i^k/\partial x_k = (\frac{1}{2}\partial h_{ip}/\partial x_q + \frac{1}{2}\partial h_{ip}/\partial x_q - \partial h_{pq}/\partial x_i) \cdot \overline{D}^{pq} h. \tag{23}$$

Then (21) would indeed assume the form of a differential law of conservation of energy and momentum:

$$(\partial/\partial x_k)(\mu u_i u^k + L_i^k + G_i^k) = 0. \tag{24}$$

There are 16 linearly independent tensors G_{ik} of this sort, and I have checked whether for any linear combination of them a relation like (23) can hold; the result was negative. This applies in particular to the field equations which Birkhoff adopts:

$$\overline{D}_{ik} h \equiv \Box h_{ik} - \frac{1}{2}\Box h \cdot \delta_{ik} = \sigma u_i u_k$$

(and which he interprets in a slightly different manner in terms of a fluid of peculiar nature). It may, therefore, be said that Birkhoff *sacrifices the conservation law of energy and momentum to that of mass.*

That it is possible to develop a theory of dualistic type in which the conservation law for energy-momentum holds is proved by a certain interpretation of the 'degenerate Einstein theory' (D) which I had used to illustrate (B): One starts with the field equations of (L) in the normalized form (10 and 11), sets $T_{ik} = \sigma u_i u_k$, *throws away the supplementary conditions* (10) in order to make room for an extra term in the mechanical equations (7) and finally replaces the latter not by (21), but by

$$(\partial/\partial x_k)(\mu u_i u^k + L_i^k) - \frac{\sigma}{2}\frac{\partial h_{pq}}{\partial x_i} u^p u^q = 0.$$

Of course, mass is not conservative in this set-up; one finds instead

$$\partial(\mu u^k)/\partial x_k = (\sigma/6)(\partial h_{pq}/\partial x_r + \partial h_{qr}/\partial x_p + \partial h_{rp}/\partial x_q)u^p u^q u^r.$$

But the conservation laws for energy and momentum (24) hold if one defines the gravitational energy tensor G_{ik} by

$$G_i^k = -H_i^k + \frac{1}{2}\delta_i^k \cdot H \qquad (H = H_p^p)$$

where

$$H_{ik} = \frac{1}{2}\frac{\partial h_{pq}}{\partial x_i}\frac{\partial h^{pq}}{\partial x_k} - \frac{1}{4}\frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_k}.$$

But it is not my intention to propagandize this or any other dualistic theory of gravitation!

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