The singular points of Einstein's universe †

by Marcel Brillouin

(translation by S. Antoci*)

1. Einstein's four-dimensional Universe is determined by the ten $g_{\mu\nu}$ of its ds^2. In order to determine them it is not sufficient to know the six independent partial differential equations that they must fulfil; one needs also to know the conditions at the boundaries of the Universe, which are necessary for specialising the integrals in view of given problems. These boundary conditions are of two sorts. One deals with the far away state of the Universe, completely outside the region where we wish to study the events; it is the one whose choice, still in dispute, is translated into this question: is the Universe infinite? Is it finite, although without limit? I do not bother with this here. The other one deals with the singular lines that correspond to what, from the experimental viewpoint, we call the attractive masses. In Newtonian gravitation the material point of mass m corresponds to the point of the Euclidean space where the integral of Laplace's equation becomes infinite like m/r, where r is (in the neighbourhood of this point) the distance from the material point to the point where one studies the Newtonian potential. It is this kind of singularities, characteristic of matter, that I come to consider.

We first remark that, in the present state of our experimental knowledge, nothing entitles us to suppose that singular points (in four dimensions) may exist in the Universe. From the analytical viewpoint, this impossibility is evidently connected with the distinction that exists between one of the variables and the remaining three, which allows for the sign changes of ds^2, like in acoustics. It would be interesting to give precision to this remark.

2. Let us consider a static, permanent state, *i.e.* a state in which the $g_{\mu\nu}$ depend only on three of the variables, x_1, x_2, x_3, and are independent of that x_4 whose g_{44} in essentially positive. The simplest singular line of Universe is the one which, in the section with the three dimensions x_1, x_2, x_3 that we call *space*, corresponds to an *isotropic singular point*.

For a Universe that contains only one line of this kind Schwarzschild has integrated, in 1916, the differential equations that rule the $g_{\mu\nu}$, and he obtained a ds^2 that, by changing Schwarzschild's notations, I write under the form

$$ds^2 = \gamma c^2 d\tau^2 - \frac{1}{\gamma} dR^2 - (R+2m)^2(d\theta^2 + sin^2\theta d\varphi^2),$$

$$\gamma = \frac{R}{R+2m},$$

c, velocity of light (universal constant);
θ, φ, the spherical polar angles in space;
R, a length such that the sphere, whose centre is at the singular point in the (non Euclidean) space R, θ, φ, and which has R for co-ordinate radius, has a total surface equal to $4\pi(R+2m)^2$, and $2\pi(R+2m)$ as circumference of the great circle.

The function γ is positive, and equal to 1 at a large distance; m is a positive constant. If R is not zero, γ is finite and nonvanishing; there is no singularity either of $g_{\mu\nu}$ or of ds^2.

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But, if R is zero, γ is zero, and the coefficient of dR^2 is infinite: there is a pointlike singularity in this point in space, and a line singularity in the Universe.

3. One may wonder whether this singularity *limits* the Universe, and one must stop at $R = 0$ or, on the contrary, it only *traverses* the Universe, which shall continue on the other side, for $R < 0$. In the discussions at the "Collège de France", in particular in the ones held during the Easter of 1922, it has been generally argued as if $R = 0$ would mean a *catastrophic* region that one needs to cross in order to attain the true, singular limit, that one only reaches when γ is infinite, with $R = -2m$. *In my opinion, it is the first singularity, reached when $R = 0$, $\gamma = 0$ ($m > 0$), the one that limits the Universe* and that must not be crossed [*C. R.*, **175**, (November 27th 1922)].

The reason for this is peremptory, although up to now I have neglected to put it in evidence: *for $R < 0$, $\gamma < 0$, in no way the ds^2 any longer corresponds to the problem one aimed at dealing with.*

In order to see it clearly, let us put anew the letters $x_1 ... x_4$ whose physical meaning shall not be suggested by old habits. Schwarzschild's ds^2 corresponds to the following analytical problem: the $g_{\mu\nu}$ depend only on the single variable x_1; two more variables, θ and φ, enter in the manner that corresponds to the spatial isotropy around one point, and one has:

$$ds^2 = \gamma c^2 dx_4^2 - \frac{1}{\gamma} dx_1^2 - (x_1 + 2m)^2 (d\theta^2 + sin^2\theta d\varphi^2),$$

$$\gamma = \frac{x_1}{x_1 + 2m} \qquad m > 0.$$

The term γ is positive either when

$$x_1 > 0 \ \ and \ \ x_1 + 2m > 0$$

or when

$$x_1 < 0 \ \ and \ \ x_1 + 2m < 0;$$

one has truly solved the proposed problem, x_1 is a spatial variable and x_4 is a time variable.

4. *If γ is instead negative, $-2m < x_1 < 0$, the characters of length and of duration are exchanged between x_1 and x_4*; in fact now the term $-(1/\gamma)dx_1^2$ is positive, while the term $\gamma c^2 dx_4^2$ is negative.

Let us make this character visible by substituting the notation t for $-x_1$:

$$ds^2 = \frac{2m - t}{t} dt^2 - \frac{t}{2m - t} dx_4^2 - (2m - t)^2 (d\theta^2 + sin^2\theta d\varphi^2).$$

This is a ds^2 which has no longer any relation with the static problem that one aimed at dealing with. *The ds^2 for $x_1 < 0$ does not continue the one that is appropriate for $x_1 > 0$.* This discontinuity is by far sharper than all the ones that have been encountered up to now in the problems of mathematical physics. The frontier $x_1 = 0$, $R = 0$, is really an insurmountable one.

While discussing Schwarzschild's integration one notices that an arbitrary factor C_4 could have been left in the term with dx_4, and that this factor is taken equal to 1 in order that the Universe become Euclidean when x_1 is infinite. No similar condition can be imposed in the interval $-2m < x_1 < 0$; but C_4 is a real constant, and it can only be taken with a positive value. In fact, if it were negative, the ds^2 would no longer have any meaning that refer to an Einstein's universe.

5. The conclusion seems to me unescapable: the limit $R = 0$ is insurmountable; it embodies the *material* singularity.

The distance r from this origin ($R = 0$) to a point with co-ordinate radius R, calculated along a radius vector ($\theta = const., \varphi = const.$) is [1]

$$r = \sqrt{R(R+2m)} + m \ln \frac{R + m + \sqrt{R(R+2m)}}{m}.$$

The ratio between the circumference $2\pi(R+2m)$ and the radius is everywhere larger than 2π; in particular at the origin ($r = 0, R = 0$) this ratio becomes infinite. The circumference of the great circle of the origin is $4\pi m$; the spherical surface of that point has the finite value $4\pi(2m)^2$. It is this singularity that constitutes what physics calls *the material point*; it is the factor m appearing in it that must be called mass.

In view of this occurrence, the word *material point* is perhaps ill chosen. In fact, due to the finite extension of the spherical surface of the point, the variations of θ and of φ truly displace the extremity of the radius vector (of zero length) over this surface as it would occur over any other sphere whose co-ordinate radius R and whose radius r were nonvanishing.

Anyway, since nothing more pointlike can be found in Einstein's Universe, and since one really needs attaining a definition of the elementary *material test body* which, according to Einstein, follows a geodesic of the Universe to which it belongs, I will maintain this *abridged nomenclature, material point*, without forgetting its imperfection.

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(1) *Erratum.* In the already cited note of the *C. R.* the coefficient $-m/2$ of the logarithm is incorrect.