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On the Theory of Gravitation

By Hermann Weyl.

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Abstract.

A. Appendix to General Relativity.

1. The derivation of a Hamiltonian Principle for sovling problems of the mass-point given the current state of our knowledge.
2. The determination of the energy-momentum tensor from a Hamiltonian Principle for the variations of the parameters, which are due to the infinitesimal deformation of the four-dimensional spacetime continuum, if the parameters are dragged along the deformation.
3. Principles of experiment and theory. The derivation of Fermat's Principle of least time for light rays in a static gravitational field, and of an analogous principle for the path of a charged mass-point under the influence of gravitation and electricity.

B. Theory of the static, axial symmetric field.

4. Easy derivation of the Schwarzschild solution for a mass-point and transformation to another set of coordinates for the following important coordinate system: for the electrostatic and gravitational field of a charged mass-point.
5. With the construction of a particular coordinate system of the uniquely derived canonical cylindrical coordinates it is possible to calculate the fields of resting masses which have rotational symmetry, as easily by this method as with Newton's theory; between the solutions of Newton and Einstein there is a relation described by elementary functions.
6. The same is true for the electrostatic and gravitational field of charged mass-points with rotational symmetry.

A. Appendix to the Theory of Gravitation.

1. Hamiltonian Principle.

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Hilbert[3], following Mie's theory[4], in a more general way then Lorentz[5] and the creator of the theory of gravitation himself[6], showed that gravitation equations can actually be derived by a Hamiltonian principle. His first formulation was not successful because we don't know the Hamiltonain function for the matter (?) since we don't even know how to describe it with independent parameters.

Under these circumstances it seems to me to be important to derive a Hamiltonian principle which includes our current knowledge of matter, in the sense of Einstein, i.e. the energy-momentum. From this principle, which looks different to all formulations so far, the following laws should come from the same source.

1. The inhomogeneous gravitational equation which says that the energy-momentum tensor causes the curvature of spacetime. The energy-momentum tensor will only be construicted from the energy-momentum tensor of the other, and from the kinetic energy-momentum tensor of the matter in the sense of $\rho u_i u_k$, in which the invariant matter density appears in the components u^i $(i = 1, 2, 3, 4)$ of the four-velocity. I do not take into account the constitution of matter or its cohesive forces.
2. The Maxwell-Lorentz equations, which have the same meaning in the electron theory, the only electric current is the displacement current.
3. The law for the ponderomotive forces in the electromagnetic field and the mechanical equations which describe the movement of those masses in the influence of those forces and the gravitational field.

If the x_i are the four coordinates of spacetime[7]

$$g_{ik} dx_i dx_k \tag{1}$$

is the invariant quadratic differential form (??? index 3)[8] whose coefficients are the gravitational potentials, and $\phi_i dx_i$ the invariant linear differential form whose coefficients ϕ_i are the components of the electromagnetic four-potential. The indefinite integral

$$-\frac{1}{2}\int H d\omega \qquad where \qquad H = g^{ik}\left(\left\{{ik \atop r}\right\}\left\{{rs \atop s}\right\} - \left\{{ir \atop s}\right\}\left\{{ks \atop r}\right\}\right)$$

I call, in this part of spacetime, the influence of the gravitational field, and the integral

$$-\frac{1}{2}\int L d\omega \qquad where \qquad L = \frac{1}{2}F_{ik}F^{ik} = \frac{1}{2}g^{ij}g^{kh}F_{ik}F_{jh}$$

[3]Gott. Nachr. 1915, Sitzung vol. **20**, November

[4]Ann. d. Phys. **37**, p. 511, **39**, p.1, 1912; **40**, p. 1, 1913.

[5]3st ref

[6]A. Einstein, sitzungsber. d. Preuss. Akad. d. Wiss., **42**, p. 1111, 1916.

[7]ref 2

[8]Each quadratic form can be transformed linearly to sums and differences of squares; the number of the negative elements is called ????. That this index is unique is part of the law of the quadratic forms.

is the influence of the electromagnetic field. Here

$$F_{ik} = \frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_i}{\partial x_k}$$

are the components of the electromagnetic field, and dw is the four-dimensional elemental volume

$$\sqrt{g} dx_1 dx_2 dx_3 dx_4, \qquad -g = \det |g_{ik}|.$$

In this phenomenological theory the field is opposed by the "substance" consisting of a three-dimensional moving continuum which one can think of in a (mathematical) way in infinitesimal elements. For each element there is a certain unvarying mass dm and an unvarying electrical charge de; it corresponds to the worldline whose direction is given by the proportion of the differentials $dx_1 : dx_2 : dx_3 : dx_4$. The value of

$$\int \left\{ dm \int \sqrt{g_{ik} dx_i dx_k} \right\}, \tag{2}$$

in which the outer integral is over the whole substance, but the inner integral is over the part of the worldline of the substance element dm which is inside the world region $\wp$, and I call this the gravitational field due to the substance. We assume that the movement of the substance is related to the gravitational field in such a way that the square root which appears in the inner integral, which is the proper time, is always positive. We transform (2) into an integral $\int \rho d\omega$ over the world region $\wp$, in which r is the invariant spacetime function which is the absolute mass density. Analogously to (2) the integral for the influence on the substance due to the electric field is

$$\int \left\{ de \int \phi_i dx_i \right\};$$

and the absolute electric charge density ε is defined by

$$\int_{\wp} \varepsilon d\omega = \int \left\{ de \int ds \right\}.$$

The Hamiltonian principle is:

The sum of the field and the influence of the gravitation and the electricity is in each world region an extremum opposed to any variation on the border of the electromagnetic and gravitational fields and of the time and space displacements of the moving substance[9].

The variation of the g^{ik} (the electromagnetic field has no variation and the substance has the same worldline) derives the Einstein gravitational equations (1). Variation of the electromagnetic potential ϕ_i gives the Maxwell-Lorentz equations

[9]footnote

(II) $$\frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}F^{ik}\right)}{\partial x_k} = J^i = \varepsilon\frac{dx_i}{ds},$$

and the variation of the worldlines of the substance element firstly gives the ??? equations

(III) $$\rho\left(\frac{d^2x_i}{ds^2} + \left\{ {hk \atop i} \right\}\frac{dx_h}{ds}\frac{dx_i}{ds}\right) = p^i,$$

where p^i is the contravariant component of the force; its covariant is given by

$$p_i = F_{ik}J^k.$$

Of course, these laws are not independent of each other. The mechanical equation (III), together with the conservation of matter, is a mathematical result of the laws (I) and (II).

2. The Energy-Momentum Tensor

From the phenomenological theory I go to a theory that can be formulated. With the authors quoted above the world is dominated by a principle of the following form

$$\int_{\wp}(H - M)\,d\omega = \ Extremum.$$

The world density M of the cause of the material action is a universal function of the independent parameters which characterise the action. It's derivative, the first or even of higher order, ofter the coordinates x_i and the g^{ij}. To give an example, M in Mie's theory depends not only on g^{ik} but also on the four components ϕ_i of the electromagnetic potential and the field components F_{ik} which are derived from the ϕ_i by differentiation. The derivation of the mechanical equations in the phenomenological theory gives me the idea that in general the principle of conservation of energy and momentum is the expression that the Hamiltonian Principle is true, especially for those infinitesimally small variations which are caused by the infinitesimal deformation in a such a way that the parameters are dragged along with this deformation. Indeed, this is the case and it looks like this is going to be the easiest and natural derivation of the energy principle.

If we put $M\sqrt{g} = \aleph$, so that the energy-momentum tensor T_{ik} is defined with the equation for the total differential for $\aleph$:

$$\frac{1}{\sqrt{g}}\delta\aleph = -T_{ik}\delta g^{ik} + \frac{1}{\sqrt{g}}\left(\delta\aleph\right)_0,$$

where $\left(\delta\aleph\right)_0$, stands for those terms which have the differentials of the material parameters, (for example the ϕ_i and F_{ik}) in linear form. With a coordinate transformation of the form

$$\overline{x}_i = \overline{x}_i(x_1, x_2, x_3.x_4)$$

the transformation of the contravariant tensor g^{ik} is given by

$$\overline{g}^{ik} = g^{\alpha\beta} \frac{\partial \overline{x}_i}{\partial x_\alpha} \frac{\partial \overline{x}_k}{\partial x_\beta}.$$

If the transformation is infinitesimal:

$$\overline{x}_i = x_i + \varepsilon \cdot \xi_i \left(x_1, x_2, x_3, x_4\right)$$

(ε describes the infinitesimal, this means ε converges to zero), as a result we get for the difference

$$\overline{g}^{ik}(\overline{x}) - g^{ik}(x) = \delta g^{ik}$$

the value for g^{ik} and $\overline{g}^{ik}$ for two systems (x) and $(\overline{x})$, which describe in the old and the new coordinate systems the same world point

$$\delta g^{ik} = \varepsilon \left(g^{\alpha k} \frac{\partial \xi_i}{\partial x_\alpha} + g^{i\beta} \frac{\partial \xi_k}{\partial x_\beta} \right).$$

If we do the same for the parameters of the material action, so we get, if we state that with such an infinitesimally small transformation of the invariant M stays the same, the law that we get shows that the energy-momentum tensor is dependent on g^{ik} and the material parameters.

We look at a world region $\wp$, which is described by the coordinates x_i, a certain mathematical area $\Im$ which is in the area of the variables x^i. If the infinitesimal transformation has the property that the variation ξ_i on the boundary of the region $\wp$ disappears with its derivatives, so that the world region $\Im$ in the new coordinates $\overline{x}_i$ is the same mathematical region $\Im$. I put

$$\Delta g^{ik} = \overline{g}^{ik}(x) - g^{ik}(x) = \delta g^{ik} + \left\{ \overline{g}^{ik}(x) - \overline{g}^{ik}(\overline{x}) \right\}$$

$$= \delta g^{ik} - \varepsilon \cdot \frac{\partial g^{ik}}{\partial x_\alpha} \xi_\alpha,$$

so I make the difference from g^{ik} and $\overline{g}^{ik}$ on two spacetime points, in which the second in the new coordinate system, the same coordinate values has as the first are in the old system. In other words, I make a virtual displacement. The same meaning of Δ is the same for all other values. If I write in short dx for the integration element $dx_1 dx_2 dx_3 dx_4$ so $\int \aleph dx$ is an invariant, then

$$\int_\Im \aleph dx = \int_\Im \overline{\aleph}(\overline{x}) d\overline{x} = \int_\Im \overline{\aleph}(x) dx; \quad \textit{furthermore} \quad \int_\Im \Delta\aleph \cdot dx = 0.$$

But it is

$$\Delta\aleph = -L_{ik} \Delta g^{ik} + \left(\Delta\aleph\right)_0 \qquad \left(L_{ik} = \sqrt{g} \cdot T_{ik}\right).$$

For the following you must be aware that: in the transformed coordinate system - for example, I take Mie's theory - there are the same form of the equations,

$$\frac{\partial \overline{\phi}_k(\overline{x})}{\partial \overline{x}_i} - \frac{\partial \overline{\phi}_i(\overline{x})}{\partial \overline{x}_k} = \overline{F}_{ik}(\overline{x}),$$

and since it doesn't matter what we call the coordinates, we can write

$$\frac{\partial\overline{\phi}_k(x)}{\partial x_i} - \frac{\partial\overline{\phi}_i(x)}{\partial x_k} = \overline{F}_{ik}(x).$$

So the relations

$$\frac{\partial\phi_k}{\partial x_i} - \frac{\partial\phi_i}{\partial x_k} = F_{ik}$$

stay the same. If we go from the functions ϕ_i, F_{ik} to the functions $\overline{\phi}_i$, $\overline{F}_{ik}$ of the same coordinate x_i, this means they stay the same under the variation $\varDelta$ (but not under the variation δ). Therefore, with the general principle of (cause ??), in which we leave the g^{ik} unmodified, this means, with the laws of the material action it is therefore

$$\int_{\Im} (\varDelta\aleph)_0 \, dx = 0, \qquad \textit{and so} \qquad \int L_{ik}\varDelta g^{ik} \cdot dx = 0.$$

If we use the expression for $\varDelta g^{ik}$ and eliminate the derivative of the displacement component with the help of partial integration, we get

$$\int \left\{ \frac{\partial L_i^k}{\partial x_k} + \frac{1}{2}\frac{\partial g^{rs}}{\partial x_i} L_{rs} \right\} \xi_i dx = 0,$$

and with that it is proved for the energy-momentum equations,

$$\frac{\partial L_i^k}{\partial x_k} + \frac{1}{2}\frac{\partial g^{rs}}{\partial x_i} L_{rs} = 0, \tag{3}$$

For the variation of the gravitational field which disappears on the boundary of the world region $\wp$, it is

$$\delta \int H d\omega = \int \left(R_{ik} - \frac{1}{2} g_{ik} R \right) \delta g^{ik} \cdot d\omega;$$

whereas R_{ik}, which is the Riemann curvature tensor, and the invariant

$$R = g^{ik} R_{ik}.$$

If we use the same ideas on H instead of on M (that is H is also a differential quotient which consists of the g^{ik}, it doesn't matter) we find without any calculations that the tensor

$$R_{ik} - \frac{1}{2} g_{ik} R,$$

Put in place of T_{ik} solves the same equation (3) identically. The energy-momentum tensor is therefore not only a mathematical result of the laws of the material action but also of the gravitational equations

$$R_{ik} - \frac{1}{2} g_{ik} R = -T_{ik}.$$

Instead of the old separation of geometry, mechanics and physics in the Einstein theory there is the opposing of the material action and the gravitation. Because of the presence of the energy-momentum tensor, on one hand is a result of the laws of the material action, on the other hand the necessary consequence that the matter describes how matter is part of the gravitation equation. In the system of the material and the gravitational laws there are four unnecessary equations. In the general solution there must be four arbitrary functions since the equations, because of their invariant nature, the coordinate system of the x_i is undetermined.[10]

3. Connexion with observations. Light rays and path of a particle in the static gravitational field.

We can only capture the "objective" world, which physics is striving to peel out of the lived reality, after its designated content with mathematical concepts. But in order to signify the meaning, which this mathematical concept possess for reality, a task for the "Perception" -theory, which is naturally not only given by the physical concepts alone, but also has to be afforded to the steady vocation of the visual experiences in the consciousness. Of this kind is the connexion between the vibrations frequency and the quality of the sense for colour. In general it seems that the impinging upon the "receptors" by the energy-momentum-current is responsible due to its intensity, through the way of its spacetime variability, for the quality of the reception. Here I would like to describe this manner for a rarely simple relation for object and subject.

We think of moving, single, light emitting point-mass particles in the four-dimensional physical world, namely the star. For the sake of simplicity we use geometric optics where the worldlines of the light emitted by the stars are geodesics. In general the equations, with the use of a suitable parameter s, of a geodesic worldline are:

$$\frac{d^2 x_i}{ds^2} + \left\{ {kh \atop i} \right\} \frac{dx_k}{ds} \frac{dx_h}{ds} = 0. \tag{4}$$

From this you obtain

$$F \equiv g_{ik} \frac{dx_i}{ds} \frac{dx_k}{ds} = const.$$

The singular geodesic worldlines are recognized in that way that for them the constant results in zero (while for the worldlines for mass particles the constant is positive). We simplify the consciousness, the "Monad", to a "point-like eye". In each moment of its life it takes up a certain point of the spacetime, it describes a worldline. Those points of the worldline it experiences them timely as step-by-step. We concentrate on a certain moment; at the point P, in which it takes up the "Monad", the potentials for the gravitation may have the values g_{ik}; the dx_i are supposed to be components of the element e of its worldline,

[10]footnote

the ratio of the dx_i indicate their world-direction (velocity). We have to assume that the direction is timelike, so that for them it results in

$$ds^2 = g_{ik} dx_i dx_k > 0.$$

Instead of writing dx_i for the derivatives I will use x_i since all our observations refer to the same point P.

Two line-elements x_i, x'_i are orthogonal if

$$g_{ik} x_i x'_k = 0.$$

A priori I claim that all (from P outgoing) line-elements, which are orthogonal to the timelike e, are on their part spacelike, so that they stretch an infinitesimally small three-dimension region $\Re$, which is imposed a positive-definite measure by the form $-ds^2$. The "Monad" experiences this region $\Re$ as its immediate "special environment". In order to prove our "claim" we assume e to be the fourth coordinate axis; then the first three components of e are equal to zero and $g_{44} > 0$. Now, we can put

$$ds^2 = \Sigma_{i,k=1}^4 \, g_{ik} x_i x_k = g_{44} \left(x_4 + \frac{g_{14}}{g_{44}} x_1 + \frac{g_{24}}{g_{44}} x_2 + \frac{g_{34}}{g_{44}} x_3 \right)^2 - \mathit{quadr.}\ F.(x_1\ x_2\ x_3).$$

If we introduce

$$x_4 + \frac{g_{14}}{g_{44}} x_1 + \frac{g_{24}}{g_{44}} x_2 + \frac{g_{34}}{g_{44}} x_3$$

as the fourth coordinate in place of the x_4 so far, we get

$$ds^2 = g_{44} x_4^2 - Q(x_1\ x_2\ x_3).$$

Because ds^2 has the signature 3, the quadratic form Q has to be positive-definite. All and only those elements for which $x_4 = 0$ are orthogonal to e. With that we have proved our "claim". Furthermore, we are able to see that each line-element can be split up in a unique way in two terms of a sum where one is parallel to e (with components proportional to e) and the other is perpendicular to e. That second term we label as the "space-direction" of the line-element. Various space-directions, which are perpendicular to e, form an angle that can be determined in the usual way with the quadratic form $-ds^2$ which is positive for those. The so determined angle of the space-direction of the worldlines of two light signals that arrive in the point P from two different stars is identified with the angle between the two directions (in a visual concept) in which a point eye catches sight of those two stars at that moment. We take theses differences in direction as at least approximate, straight and visually detectable; indeed in seeing there isn't only a quality aspect given by the perception but also this quality as a special extension (a moment that cannot be reduced to the base of the perception in any kind). With the use of suitable instrumentation for observation, the determination of the angular distribution is more accurate; whereas the only power for consciousness is left in the distinguishing or undistinguishing of two directions (such as alignment of cross-hairs and stars, reading the scale of the semi-circle). This easy scheme is enough in order to describe the principle ways in which the observations of stars can be used for checking of Einstein's theory.

In connexion with the previous I would like to show how easy it is to derive the Fermat Principle of least time from the general principle "The worldline of a light signal is a geodesic line" in the case of a static gravitational field. We choose the parameter s for the representation of the geodesic line in a way that corresponds to equation (4), and so it is characterized by the variational principle

$$\delta \int F ds = 0, \tag{5}$$

true for a virtual displacement for which the endpoints of the considered world line-element stay fixed. Except for the singular worldlines it is possible to consider the equation

$$\delta \int \sqrt{F} ds = 0.$$

In the static case we put $x_4 = t$; the quadratic formula (1) has the form

$$f dt^2 - d\sigma^2,$$

where ds^2 is a quadratic form of the space derivatives dx_1, dx_2, dx_3, whose coefficients are, just as f, the square of the velocity of light, independent of the time t. In this case, if we only vary t,

$$\delta \int F ds = 2 \int f \frac{dt}{ds} d\delta t = \left[2f \frac{dt}{ds} \delta t \right] - 2 \int \frac{d}{ds} \left(f \frac{dt}{ds} \right) \delta t ds. \tag{6}$$

Therefore,

$$f \frac{dt}{ds} = const. = E.$$

If we drop the condition that except $\delta x_1, \delta x_2, \delta x_3, \delta t$ also vanishes on the boundary of the integration, we have to replace (5), as arises from (6), with

$$\delta \int F ds = [2E\delta t] = 2\delta \int E dt. \tag{7}$$

When we vary the special trajectory of the light signal at will, holding the ends steady, and imagine however the varied curve also being traversed with the speed of light, thus applies for the original as well as the varied curve,

$$F = 0, \qquad d\sigma = \sqrt{f} \cdot dt,$$

And (7) becomes

$$\delta \int dt = 0 \qquad or \qquad \delta \int \frac{d\sigma}{\sqrt{f}} = 0,$$

which gives the Principle of Fermat. Now the time has been completely eliminated; the last formulation refers alone to the spatial course of the ray of light and applies only to each piece of it, if this is varied at will with the holding of the beginning and end points. We can apply the same method in order to derive a minimal principle for the trajectory of a mass point particle in the static gravitational field. We assume immediately that the point with mass m also carries the electric charge e and is exposed to an electric field with potential Φ. By section **1** the variation principle reads, if ds is meant to be the derivative of the proper time,

$$\delta\left\{m\int ds + e\int \Phi dt\right\} = 0. \tag{8}$$

If we vary only t, not the spatial coordinates, the left side is,

$$= \int \left\{mf\frac{dt}{ds} + e\Phi\right\} d\delta t.$$

Thus

$$mf\frac{dt}{ds} + e\Phi = const. = E, \tag{9}$$

And the variational principle (8) has to be replaced, if we give up the condition that except $\delta x_1, \delta x_2, \delta x_3, \delta t$ also vanishes on the boundary of the integration, with

$$\delta\left\{m\int ds + e\int \Phi dt\right\} = [E\delta t] = \delta\int Edt. \tag{10}$$

Introducing the value

$$ds = \sqrt{fdt^2 - d\sigma^2}$$

into (9), and setting the abbreviation,

$$U = \frac{E - e\Phi}{\sqrt{f}},$$

arises the law of the velocity,

$$\frac{Ud\sigma}{\sqrt{f\left(U^2 - m^2\right)}} = dt. \tag{11}$$

When we think of the spatial varied trajectory being traversed, while holding the ends steady, in particular with the same law for the velocity as the initial curve, then (9) is also valid for the varied curve. Therefore, we get from (10):

$$\delta\int\left\{\frac{m^2 f}{E - e\Phi} - (E - e\Phi)\right\} dt = \delta\int \frac{\sqrt{f}\left(m^2 - U^2\right)}{U} dt = 0.$$

In that case we can use expression (II) for dt, since this equation holds for all variations as assumed; thus, the time is eliminated completely and we find that the spatial trajectory is characterized by the minimal principle[11],

$$\delta\int \sqrt{U^2 - m^2} d\sigma = 0.$$

[11] Levi-Civita

B. Theory of the static, rotationally symmetric gravitational field.

4. Point-mass with and without electric charge.

For the following it is necessary to make some remarks about the Schwarzschild gravitational field of the motionless point-mass. A three-dimensional spherically symmetric line element has, in suitable coordinates, the necessary form

$$d\sigma^2 = \mu(dx_1^2 + dx_2^2 + dx_3^2) + l(xdx_1 + xdx_2 + xdx_3)^2,$$

where μ and l are dependent only on the distance

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The scale, in which this distance is measured, can be transformed in such a way that $\mu = 1$; which it will take hereinafter. For the four-dimensional line element we have to make

$$ds^s = f dx_4^2 - d\sigma^2,$$

where f is a function only of r as well. If we put

$$1 + ir^2 = h$$

and the square root of the determinate $hf = \omega$, so with a short calculation, which we do properly for the point $x_1 = r, x_2 = 0, x_3 = 0$, we obtain

$$H = g^{ik}\left(\left\{{ik \atop r}\right\}\left\{{rs \atop s}\right\} - \left\{{ir \atop s}\right\}\left\{{ks \atop r}\right\}\right) \qquad \textit{and the value} \;\; -\frac{2lr}{h}\cdot\frac{\omega'}{\omega}.$$

The prime indicates differentiation with respect to r. Furthermore, let

$$-\frac{lr^3}{h} = \left(\frac{1}{h} - 1\right) r = v;$$

then you have to solve the variation problem

$$\delta\int v\omega' dr = 0 \qquad or \qquad \delta\int \omega v' dr = 0;$$

where v and ω may be treated as the independent varying functions. Variation in v results in

$$\omega' = 0, \qquad \omega = const.$$

and with the proper transformation over the still arbitrary scale unit time: $\omega = 1$. Variation of ω results in

$$v' = 0, \qquad v = const. = -2a;$$

$$f = \frac{1}{h} = 1 - \frac{2a}{r}.$$

a is linked with the mass m through the equation $a = \kappa m$; we call a the gravitational radius of the mass m.

In order to understand better the geometry of the line element ds^2 we restrict ourselves to the plane surface which passes through the equator $x^3 = 0$. If we introduce polar coordinates

$$x_1 = r, \qquad x_2 = r\sin\theta,$$

we get

$$d\sigma^2 = hdr^2 + r^2 d\theta^2.$$

This line-element characterizes the geometry, which is valid for the following rotation ellipsoid in Euclidean space with orthogonal coordinates x_1, x_2, z:

$$z = \sqrt{8a\,(r - 2a)},$$

if the same refers to the polar coordinates r, θ, passing through orthogonal projections on the plane $z = 0$. The projection covers the outer part of the sphere $r \geq 2a$ twice and the inner part not at all. So with natural analytic continuation, the real space, with the use of coordinates x_i, is covered twice in the region represented by $r = 2a$. The two overlapping regions are separated by the sphere $r = 2a$, on which is located the mass, and the determination of the mass becomes a singularity, and you will have to call the two halves the outer and the inner of the mass-point.

Perhaps this is more obvious with the introduction of a different coordinate system, by which I will have to transform the Schwarzschild equations anyway, in order to expand further. The transformation equations are

$$x_1' = \frac{r'}{r}x_1,\ x_2' = \frac{r'}{r}x_2,\ x_3' = \frac{r'}{r}x_3;\ r = \left(r' + \frac{a}{2}\right)^2 \cdot \frac{1}{r'}.$$

If I drop the prime after the transformation, I obtain

$$d\sigma^2 = \left(1 + \frac{a}{2r}\right)^4 \left(dx_1^2 + dx_2^2 + dx_3^2\right),\ f = \left(\frac{r - a/2}{r + a/2}\right)^2. \tag{12}$$

So in the new coordinates the line element of the gravitational space is conformal to the Euclidean; the linear enlargement proportion is

$$\left(1 + \frac{a}{2r}\right)^2.$$

$d\sigma^2$ is regular for all values $r > 0$, f is always positive and only becomes zero when

$$r = \frac{a}{2}.$$

The circumference of the circle $x_1^2 + x_2^2 = r^2$ is

$$2\pi r\left(1 + \frac{a}{2r}\right)^2;$$

this function decreases monotonically, if we let r go from $+\infty$ to lower values, the value $4\pi a$, which is reached when

$$r = \frac{a}{2}$$

then when r decreases to zero, it starts increasing again and increases over all limits (borders). With the conception above, the region

$$r > \frac{a}{2},$$

corresponds to the outer

$$r < \frac{a}{2},$$

to the limit of the mass point. With analytical continuation

$$\sqrt{f} = \frac{r - a/2}{r + a/2}$$

becomes negative in the inner part, so the cosmic time and the eigentime are counter moving for a motionless point in the inner region. (Of course, in Nature, only the singularity not reaching part of the solution can be realised.)

If the mass-point carries an electric charge and ϕ is the electric potential, then by the Lagrangian principle gives, in the CGS-system,

$$\delta \int \left(v\omega' + \frac{x}{c^2} \frac{\Phi'^2 r^2}{\omega} \right) dr = 0.$$

The variation on v gives

$$\omega' = 0, \qquad \omega = const. = 1.$$

Variation of ϕ gives

$$\frac{d}{rd} \left(\frac{r^2 \Phi'}{\omega} \right) = 0, \qquad \Rightarrow \Phi = \frac{e}{r}.$$

Thus, for the electrostatic potential we obtain the same equation as that without the consideration of gravitation. The constant e is the electric charge (in the usual electrostatic scale). But with the variation of ω we obtain

$$v' + \frac{x}{c^2} \frac{\Phi'^2 r^2}{\omega^2} = 0$$

and from this it follows that

$$v = -2a + \frac{x}{c^2} \frac{e^2}{r}, \qquad \frac{1}{h} = f = 1 - \frac{2a}{r} + \frac{x}{c^2} \frac{e^2}{r^2}.$$

As you can see, in f there is besides the mass dependent term $-2a/r$, also an electric term. $a = \kappa m$ is again the gravitational radius of the mass m. Analogously the length

$$a' = \frac{e\sqrt{x}}{c}$$

is called the gravitational radius of the electric charge e. If the distance r is comparable to a, the mass term is approximately 1, but if $r \approx a'$ the electric term is 1. f stays positive for all the values of r if $|a'| > a$; for an electron the ratio a'/a is of the magnitude 10^{20}. In distances which are comparable with

$$a'' = \frac{e^2}{mc^2}$$

the mass term and the electric term have, for the gravitational potential f, the same magnitude; only if r is large compared to a", the "superposition" principle is true in the way that the electrostatic potential is determined by the charge and the gravitational potential by the mass with the usual equations. Thus a", a quantity which is on different occasions called the "radius of the electron", can be called the radius of the "force" sphere. There is the relation $a' = \sqrt{aa"}$.

After having formulated the field of the mass-point carrying electric charge, we can, with the help of the last paragraph in section **3**, easily calculate the motion of a test particle under the influence of that field, whose charge and mass are negligible against the one responsible for the field; the problem is rigorously solved as in the chargeless case (motion of planets) with elliptic functions.

5.