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A Statical Smooth Extension of Schwarzschild's Metric.

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According to the problem formulation, the time co-ordinate x_0 describes the real line $\Re$ and the mass centre is the origin of $\Re^3$. Consequently, the base manifold is taken to be the topological product $\Re \times \Re^3 = \Re^4$. Let x_1, x_2, x_3 be the co-ordinates of the points $x \in \Re^3$ and let S be the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$. It is customary to consider, instead of the x_1, x_2, x_3, the polar co-ordinates,

$$\varrho = \sqrt{x_1^2 + x_2^2 + x_3^2} \in \bar{\Re}_+ = [0, +\infty[\quad \text{and} \quad u = (u_1, u_2, u_3) \in S\,,$$

and thus to replace $\Re^3$ by the manifold with boudary $\bar{\Re}_+ \times S$. to avoid the difficulties resulting from the modification of the base topology, we limit ourselves to the space $\Re \times \Re^3 = \Re^4$ for topological problems and we make use of polar co-ordinates only for calculations.

Let us now consider Weyl's expression ([1]) for the radially symmetric metrics of $\Re^3$

$$d\sigma^2 = p(|x|)dx^2 + q(|x|)(xdx)^2$$

with the abbreviations

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad dx^2 = dx_1^2 + dx_2^2 + dx_3^2, \qquad xdx = x_1dx_1 + x_2dx_2 + x_3dx_3\,.$$

Such a metric is said to be expressed in the *canonical form* if $|x| = \varrho$ is the geodesic distance between the origin and the point $x = (x_1, x_2, x_3)$ with respect to the metric itself. An easy calculation shows that this condition is verified if $p(\varrho) + \varrho^2 q(\varrho) = 1$, whence the canonical form

$$d\sigma^2 = p(|x|)dx^2 + \frac{1}{|x|^2}\left(1 - p(|x|)\right)(xdx)^2\,.$$

Let $d\omega^2$ be the metric induced on S by the natural metric dx^2 of $|Re^3$. Then $\varrho^2 p(\varrho) d\omega^2$ is the metric induced on the sphere $|x| = \varrho =$ const by the canonical form. Consequently it is convenient to intrioduce the function $g(\varrho) = \varrho\sqrt{p(\varrho)}$ and to use constantly the following expression of the canonical form:

$$d\sigma^2 = \left(\frac{g(|x|)}{|x|}\right)^2 dx^2 + \frac{1}{|x|^2}\left(1 - \left(\frac{g(|x|)}{|x|}\right)^2\right)(xdx)^2$$

with $g(|x|) > 0$ for $|x| > 0$ and $g(0) = 0$.

Now the problem of the radially symmetric statical field may be formulated as follows. Determine the field at each point $x \in \Re^3$ through the geodesic distance $|x| = \varrho$ of this point from the origin. In other words, determine the space-time metric

$$ds^2 = \left(f(|x|)\right)^2 dx_0^2 - \left[\left(\frac{g(|x|)}{|x|}\right)^2 dx^2 + \frac{1}{|x|^2}\left(1 - \left(\frac{g(|x|)}{|x|}\right)^2\right)(xdx)^2\right]$$

[1] H. WEYL: *Space, Time, Matter* (New York, N.Y.).

satisfying the equations of gravitation.

Introducing polar co-ordinates for the sake of calculation convenience, we have $xdx = \varrho d\varrho$, $dx^2 = d\varrho^2 + \varrho^2 d\omega^2$, whence

$$ds^2 = (f(\varrho))^2\, dx_0^2 - \left(d\varrho^2 + (g(\varrho))^2 d\omega^2\right) .$$

The equations of gravitation outside the mass are now established as follows:

$$2\frac{g''}{g} + \frac{g'^2}{g^2} - \frac{1}{g^2} = 0, \quad 2\frac{f'g'}{fg} + \frac{g'^2}{g^2} - \frac{1}{g^2} = 0, \quad \frac{f''}{f} + \frac{f'g'}{fg} + \frac{g''}{g} = 0,$$

the remaining equations being verified identically. The first two give $g'' = f'g'/f$, that is $f(\varrho) = g'(\varrho)$. In other words *the light velocity is equal to the derivative of the curvature radius of the spheres* $\varrho =$ const *considered with the induced metric.*

Now the first equation gives $g - gg'^2 = A =$ const or $dg/d\varrho = \sqrt{1 - A/g}$, which leads to the determination of $g(\varrho)$ by means of the equation

$$\varrho = \text{ const } + \int \frac{dg}{\sqrt{1 - A/g}} = C_1 + A \log\left(\sqrt{g} + \sqrt{g - A}\right) + \sqrt{g(g - A)} .$$

It is easy to see that the functions $g(\varrho)$ and $f(\varrho) = g'(\varrho) = \sqrt{1 - A/g(\varrho)}$ thus obtained satisfy the third equation. Hence it only remains to determine the constants A and C_1. We first remark that $F(g) = C_1 + A \log\left(\sqrt{g} + \sqrt{g - A}\right) + \sqrt{g(g - A)}$ is strictly increasing function convergent to $+\infty$ as $g \to +\infty$, hence its inverse $g(\varrho)$ converges also to $+\infty$ as $\varrho \to +\infty$. It follows that

$$\frac{\varrho}{g(\varrho)} = 1 + \frac{C_1}{g} + \frac{A}{2}\frac{\log g}{g} + \frac{A}{g}\log\left(1 + \sqrt{1 - \frac{A}{g}}\right) - \frac{A}{g(1 + \sqrt{1 - A/g})}$$

converges to 1 as $\varrho \to +\infty$ and the coefficient $(f(\varrho))^2 = 1 - A/g$ of dx_0^2 can be written $1 - A/g - A\epsilon(\varrho)/\varrho$ with $\lim_{\varrho \to +\infty} \epsilon(\varrho) = 0$. The well-known approximation for great distances requires $A = 2km/c^2 = 2\mu$. On the other hand, returning to the base manifold $\Re \times \Re^3 = \Re^4$, we see that the relation

$$\lim_{|x| \to +\infty} \frac{g(|x|)}{|x|} = 1$$

implies that the metric (1) converges to the pesudo-Euclidean form $dx_0^2 - dx^2$ as $|x| \to +\infty$. Substituting the original positive value $A = 2\mu$ into $F(g)$ and introducing the constant $\varrho_0 = C_1 + 2\mu \log\sqrt{2\mu}$, we find the equation

$$\varrho = \varrho_0 + 2\mu \log\left(\sqrt{\frac{g}{2\mu}} + \sqrt{\frac{g}{2\mu} - 1}\right) + \sqrt{g(g - 2\mu)}$$

which determines $g(\varrho)$. As $\sqrt{g - 2\mu}$ must be real, $g(\varrho)$ possesses the minimum 2μ obtained for $\varrho = \varrho_0$. The value ϱ_0 is not known, but it depends probably on the mass m and even on the characteristics of the matter (Fig. 1).

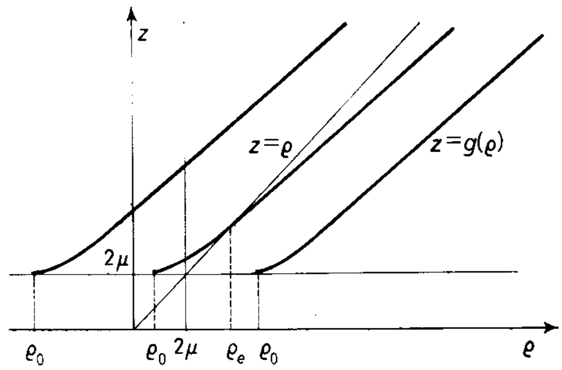

Fig. 1

1) *Suppose first that* $\varrho_0 > 0$. The obtained metric which is valid outside the mass $ds^2 = (g'(\varrho))^2 dx_0^2 - (d\varrho^2 + (g(\varrho))^2 d\omega^2)$ is defined analytically for $\varrho \geqslant \varrho_0$ and degenerates on the sphere $\varrho = \varrho_0$ because the light velocity

$$g'(\varrho) = \sqrt{1 - \frac{2\mu}{g(\rho)}}$$

is zero for $\varrho = \varrho_0$. A degenerate space-time form has no physical meaning. Consequently the domain of validity of our solution is defined by $\varrho \geqslant \varrho_1$ for some $\varrho_1 > \varrho_0$ and its extension for $\varrho_1 \leqslant \varrho_1$ must be made by taking into account the existence of matter. It is to be noticed that the difference $\varrho - g(\varrho)$ is a strictly increasing function of ϱ, hence it vanishes for a single value ϱ_e of ϱ only if $0 < \varrho_0 \leqslant 2\mu$.

2) *Suppose secondly that* $\varrho_0 \leqslant 0$. This condition contradicts the fact that $g(0)=0$. Consequently, it it will be experimentally verified, there must exist a value $\varrho_1 > 0$ such that the solution is valid only for $\varrho \geqslant \varrho_1$. It is easy to get the trajectory of a particle in the radially symmetric field by means of our solution. One must first determine the function $\varphi \to y(\varphi)$ satisfying the equation

$$E_2 \int \frac{dy}{\sqrt{2\mu E_2^2 y^3 - E_2^2 y^2 + 2\mu y + E_1^2 - 1}} = \pm\varphi \quad (E_1 = \text{ const},\ E_2 = \text{ const})$$

and then the function $\varphi \to \varrho(\varphi)$ which satisfies $g(\varrho) = 1/y(\varphi)$.

To apply this calculation to the case of the Sun, an estimate of ϱ_0 is first necessary. But the known phenomena prove that the value ϱ_0 is small in comparison with the distances coming into play. It is then legitimate to take approximately $g(\varrho) = \varrho$, and this assumption leads anew to the known values for the deflection of light rays and the advance of the perihelion of Mercury.

Remark. It is easy to verify that our solution is equivalent to Schwarzschild's one for $\varrho > \varrho_0$. In fact, on this open set the equation $r = g(\varrho)$ determines ϱ uniquely and we have $d\varrho/dr = 1/\sqrt{1 - 2\mu/r}$, whence

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) dx_0^2 - \frac{dr^2}{1 - 2\mu/r} - r^2 d\omega^2 \,.$$

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