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ON THE PRINCIPLES OF GENERAL RELATIVITY AND THE SΘ(4)-INVARIANT METRICS

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1 Introduction

The question of how gravitation propagates gives rise to two contradictory statements in general relativity: On the one hand it is asserted that every change in the distribution of matter generates a gravitational effect which is propagated in space according to the law of null geodesics. On the other hand it is asserted that, curiously, if the distribution of matter is spherically symmetrical, no gravitational waves are produced by its radial pulsations. The second assertion is supported by the Birkhoff theorem which argues that the exterior gravitational field of a non-stationary spherical body is necessarily static.

Although this discrepancy between the principles of the theory and their application to a special problem gave the relativists much trouble, everyone now believes that it is deeply rooted in the theory itself an account of Birkhoff's theorem. However general relativity is first of all a mathematical theory, so that contradictions in it cannot be allowed. Therefore we are led to ask: Is the Birkhoff theorem a true mathematical proposition? Of course, we cannot expect to answer this question by checking the computations involved in the proof of the theorem. What we have to do is to examine the underlying hypotheses, the logical structure of the proof as well as the method used.

The Birkhoff theorem, like the Schwarzschild-Droste solution to which it is closely related, was established on a vague mathematical seting. At that time the notion of manifold was not yet clarified and only local transformations of coordinates were taken into account without much attention to their domains of validity. Moreover the local character of the Einstein equations has distracted attention from the global requirements of the problems. As a consequence of this situation, the methods put forward for solving the equations of gravitation include several deficiences which continue always to persist in spite of the mathematical clarification of the theory. We emphasize specifically the abuse of the manifolds with boundaries and the abuse of the implicit transformations.

2 On the manifolds with boundaries

Several manifolds with boundaries occur in the formalism of general relativity. the most currently used is the manifold $\Re \times [0,+\infty[\times S^2$ which originates in the so-called polar coordinates. These last are correctly defined by two systems of geographical coordinates covering all S^2, thus giving rise to a C^∞ mapping of $[0,+\infty[\times S^2$ onto $\Re^3$, and, by restriction to $]0,+\infty[\times S^2$, a C^∞ diffeomorphism of $]0,+\infty[\times S^2$ onto $\Re^3 - \{(0,0,0)\}$. The inverse transformation is not defined at $(0,0,0)$, so that the polar coordinates suppress the neighbourhoods of the origin in $\Re^3$. In spite of this fact, $[0,+\infty[\times S^2$ is identified with $\Re^3$ and the meaningless term "the origin at $\rho = 0$" is commonly used. So the relativists believe that the transformation $\overline{\rho} = \rho - \alpha$, ($\alpha = const. > 0$), carries the sphere $\rho = \alpha$ into the "origin $\overline{\rho} = 0$". This extravagant idea goes back to Schwarzschild and Droste, and reappears in the definition of the so-called harmonic coordinates by K. Lanczos (1922).

Now, given a C^∞ Riemannian metric on $\Re^3$, its transform in polar coordinates is a C^∞ quadratic form on $[0,+\infty[\times S^2$, positive definite on $]0,+\infty[\times S^2$ and null on $\{0\} \times S^2$. The converse is not true. A C^∞ quadratic form on $[0,+\infty[\times S^2$ with the above peroperties may result from a Riemannian form discontinuous at $x = (0,0,0)$.

The counterpart of the preceeding statement for space-time metrics is obvious: Given a C^∞ space-time metric on $\Re^3$, its transform in polar coordinates is a C^∞ quadratic form on $\Re \times [0,+\infty[\times S^2$ which has the required signature on $\Re\times]0,+\infty[\times S^2$, but degenerates on $\Re \times \{0\} \times S^2$ in such a way that, for each $t \in \Re$, the induced metric on $\{t\} \times \{0\} \times S^2$ is null. The converse is not true. For instance, the Bondi metric

$$ds^2 = e^{2A}dt^2 + 2e^{A+B}dtd\rho - \rho^2 d\omega^2,\ (A = A(t,\rho),\ B = B(t,\rho))\,,$$

results from a uniquely defined form on $\Re \times \Re^3$:

$$ds^2 = e^{2A}dt^2 + 2e^{A+B}\ \frac{xdxdt}{||x||} - dx^2 + \frac{(xdx)^2}{||x||^2}$$

$$(\rho^2 = ||x||^2 = x_1^2 + X_2^2 + x_3^2,\ dx^2 = dx_1^2 + dx_2^2 + dx_3^2,$$
$$xdx = x_1 dx_1 + x_2 dx_2 + x_3 dx^3),$$

which presents discontinuities at $x = (0,0,0)$. As another example, the C^∞ quadratic form

$$ds^2 = dt^2 - \left(1 + \frac{\rho}{\alpha}\right)\left(d\rho^2 + \rho^2 d\omega^2\right),\ (\alpha = \text{constant positive length})\,,$$

arises from a space-time metric on $\Re \times \Re^3$:

$$ds^2 = dt^2 - \left(1 + \frac{||x||}{\alpha}\right) dx^2$$

which is everywhere continuous and C^{∞} on $\Re \times \left(\Re^3 - \{(0,0,0)\}\right)$, but not differentiable at $x=(0,0,0)$.

It follows that the current practice of formulating problems with respect to $\Re \times [0,+\infty[\times S^2$, instead of $\Re \times \Re^3$, gives rise to singularities and misleading conclusions. We emphasize that the problem must be always conceived relative to $\Re \times \Re^3$.

Of course, we cannot exhaust in this abstract the questions raised by the abuse of manifolds with boundaries in general relativity. So we confine ourselves to the following two additional remarks:

First, the space-time forms considered by the relativists on $\Re \times [0,+\infty[\times S^2$ do not always fulfil the above explained conditions of degeneracy. Moreover they frequently contain singularities incompatible with the data of the problem. In these circumstances it is impossoble to associate them with space-time metrics on $\Re \times \Re^3$ compatible with the topology of $\Re \times \Re^3$. In particular, this is the case for the solutions of Schwarzschild and Droste.

Secondly, the so-called problem of maximal extension is meaningless with respect to $\Re \times \Re^3$. Moreover the Kruskal-Szekeres maximal extension of $\Re \times [0,+\infty[\times S^2$, which is greatly appreciated by the relativists, necessitates identifications by means of discontinuous mappings, namely operations transgressing the bounds of mathematical thought.

3 On the Implicit Transformations

The initial data of every problem in general relativity contain first of all a 4-manifold defined explicitly by given systems of coordinates. In order to solve the equations of gravitation as well as for other purposes, the relativists utilize largely transformations of the initially given coordinates. These transformations may be divided into two categories according as they are explicit or implicit.

An *explicit* transformation is completely defined by known functions, so that its use is allowed provided that we confine ourselves to the domain of its validity and return finally to the initial coordinates for the verification of the boundary conditions.

An *implicit* transformation is given by means of equations containing the unknown components of the metric tensor, namely equations which cannot be actually solved. Therefore the implicit transformation is wholly indeterminate. Apart from special cases where the validity of such a transformation is required by the very formulation of the problem, the implicit transformations have no justification. However they are extensively used because of two illusive ideas:

a) *First*, it is believed that every unknown function involved in the problem can be chosen as a new coordinate. However the unknown functions are related to geometrical and physical properties and satisfy given boundary conditions, so that they cannot be reduced to coordinates. Moreover such a reduction is generally mathematically forbidden. Suppose, for instance, that an unknown component of the metric tensor is taken as a new coordinate: $y_1=g_{\alpha\beta}\left(x_0,x_1,x_2,x_3\right)$ replacing the initial coordinate x_1. Such a transformation presupposes that the derivative $\frac{\partial g_{\alpha\beta}(x_0,x_1,x_2,x_3)}{\partial x_1}$ is not zero for all values in the interval of variation of x_1. But we cannot know whether this condition is fulfilled before determining $g_{\alpha\beta}\left(x_0,x_1,x_2,x_3\right)$ by the equations of gravitation. Besides, the obtained component $g_{\alpha\beta}\left(x_0,x_1,x_2,x_3\right)$ may be such that the derivative in question vanishes for some isolated values of x_1. Then the derivative of the inverse function is $\pm\infty$ for the corresponding values of y_1, thus giving rise to singularities which are due exclusively to the introduction of an non-admissible transformation. The "horizons" and the "black holes" owe their "existence" to such inadmissible transformations

b) *Secondly*, it is believed that, if the unknowns of the problem are the 10 components of the metric tensor, then, at least in the vacuum, the system of the Einstein equations possess solutions, and that these solutions present four degrees of freedom. Neither of these assertions is proved.

The relativists assert that the Einstein-vacuum equations can be regarded as a system of 10-4 = 6 equations on account of the four contracted Bianchi identities. Nevertheless the Bianchi identies are true identities, namely satisfied whatever the components of the metric tensor may be, so that, as was pointed out by Levi-Civita, they have no effect on the equations of gravitation even if the energy-momentum tensor is not zero:

"Der Tensor $T_{\alpha\beta}$ zufolge seiner physicalischen Definition... den vier Bedingungsgleichungen genügt, die das Verschwinden der Divergenz ausdrücken; es müssten also auch die $R_{\alpha\beta}$ an entsprechenden Gleichungen gebunden sein. Wir können jedoch die Annahme einer linearen Relation zwischen den beiden Tensoren beibehalten, *ohne dass dabei eine Beziehung zwischen den $g_{\alpha\beta}$ besteht*; wir brauchen uns bloss daran zu erinneren, dass die Divergenz des Tensors

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

identisch Null ist...Setzen wir also

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = -kT_{\alpha\beta}$$

wo k eine Konstante bedeutet..., *so folgt daraus keinerlei Bindung für die $g_{\alpha\beta}$*" [3].

The relativists claim that the four degrees of freedom result not only from the Bianchi identies, but also from the covariance of the Einstein equations. So, according to A. Lichnerowicz:

"Par un choix convenable d'un système de coordonnées locales on peut astreindre quatre des potentiels à prendre localement des valeurs données ... S'il n'existait pas entre les $S_{\alpha\beta}$ un système de 4 relations, les 6 potentiels restants devraient vérifier dans le cas extérieur par exemple, 10 conditions indépendantes" [4].

In the above excerpt, it is asserted that we can find a local diffeomorphism introducing a new system of coordinates in which four components of the metric tensor are given in advance. The existence of such a diffeomorphism is questionable, but this difficulty does not regard the present discussion. We accept the existence of the diffeomorphism in question for any given expressions of the four components. Does it mean that the system of the equations $R_{\alpha\beta} = 0$ presents four degrees of freedom? Certainly not! In fact, we have an infinity of diffeomorphisms corresponding to the infinity of possible choices of for the four components. So the same solution is expressed in an infinity of systems of coordinates. No conclusion can be drawn from them about the degrees of freedom, if any, of the einstein-vacuum equations. The confusion of the number of functions defining the diffeomorphisms with the degrees of freedom is obvious.

The assertion regarding the four degrees of freedom menas, in fact, that it is possible to reduce the initial system of 6 equations by means of allowable transformations. No such reduction is known to exist. In spite of this fact, it is believed that the system of the einstein-vacuum equations needs to be completed by four "coordinate conditions". It is not surprising that the particular conditions defining the so-called harmonic coordinates lead to inconsistencies.

So far, we have assumed that the components of the metric tensor are the inknowns in the equations of gravitation. However in some cases, as for instance, when the metric tensor is endowed with symmetries, the number of unknown functions, which are invloved in the components, is less than 10, so that, if the system of the Einstein equations has solutions, it is likely that the general solution brings out several degrees of freedom. This is confirmed by the known classical solutions, although they cannot help to clarify completely the situation on account of the deeply rooted practice of removing significant functions by implicit transformations. In fact, implicit transformations have been extensively applied to the metrics with symmetries. They gave rise to poor solutions with singularities which are wrongly imputed to a "pathology of coordinates", namely to a notion empty of mathematical meaning. A brief account of the aberrations issuing from it was given in a previous paper [5]. The general conclusion that has to be drawn from the consideration of the classical solutions is that they need restating and revising.

4 SO(n)-invariant tensor fields and SΘ(4)-invariant metrics

The mathematical conception of the gravitational field generated by a spherical distribution of matter is based upon the manifold $\Re \times \Re^3 = \Re^4$ and the special role of the rotation group $SO(3)$. However, although $SO(3)$ acts naturally on $\Re^3$, it does not the same on $\Re^4$, and this is the reason why the current definition of the relevant space-time metrics is not free from logical dificulties. So, according to the explanation of Hawking and Ellis: "One might regard the essential feature of a spherically symmetric space-time as the existence of a world-line L such that the space-time is spherically symmetric about L... However, there might not exist a world-line like L in some of the space-times one would wish to regard as spherically symmetric... Thus we shall say that the space-time is spherically symmetric if it admits the group $SO(3)$ as a group of isometries, with the group orbits spacelike two-surfaces" [2]. the first application of this idea to the space-time metrics goes back to J. Eiesland [1].

Eiesland begins by considering a general space-time metric, conceived implicityly relative to $\Re \times \Re^3$:

$$ds^2 = g_{00}dx_0^2 + 2\Sigma_{i=1}^3\, g_{0i}dx_0dx_i + \Sigma_{i,j=1}^3\, g_{ij}dx_idx_j$$

and seeks to establish conditions in order that the vector fields

$$x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1},\; x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2},\; x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3},$$

be Killing vector fields with respect to the action of $SO(3)$ on the subspaces $x_0 = const$. Then the components $g_{0i}, (i = 1, 2, 3)$, cannot be taken into account. Eiesland regards them as superflous and believes that it is possible to remove them by an implicit transformation (implicit diffeomorphism) before any other operation. However, irrespective of the fact that implicit transformations are not allowed, the components in question possess a profound physical significance. By removing them, Eiseland confines himself to a truncated metric and fails to setablish the required form. Besides, he does not omit to confuse $\Re \times \Re^3$ with $\Re \times [0, +\infty[\times S^2$. No essential improvement in the conception of the problem followed Eiseland's paper because of the exclusive use of the manifold with boundary $\Re \times [0, +\infty[\times S^2$ and the systematic introduction of implicit transformations.

In order to overcome the defects of the classical conception, we have first to get rid of the restrictive view regarding specifically the metrics, and deal in general with $SO(n)$-invariant tensor fields on the oriented space $\Re^n$ for the various values of n. We then obtain complete global results without referring to implicit transformations and Killing vector fields [6].

Let Γ be the algebra of the continuous functions of $||x||$ with $x \in \Re^n$. Then we find in particular two simple and useful results:

Proposition 4.1. *If $n \geq 3$, then the set of all $SO(n)$-invariant continuous 1-forms on $\Re^n$ is the free Γ-module generated by the form $\Sigma_{i=1}^n\, x_idx_i$. This Γ-module is also $O(n)$-invariant.*

Proposition 4.2. *If $n \geq 3$, then the set of all $SO(n)$-invariant continuous covariant symmetrical tensor fields of degree 2 on $\Re^n$ is the free Γ-module generated by the two tensor fields*:

$$\Sigma_{i=1}^n\,(dx_i \otimes dx_i) \;\; and \;\; (\Sigma_{i=1}^n x_idx_i) \otimes \left(\Sigma_{j=1}^n x_jdx_j\right) .$$

This Γ-module is also $O(n)$-invariant.

Let us now return to the space $\Re \times \Re^3$. since $SO(3)$ does not act on $\Re^4$, we introduce the subgroup of $SO(4)$, denoted by $S\Theta(4)$, consisting of the matrices,

$$\begin{pmatrix} 1 & O_H \\ O_V & A \end{pmatrix} \; with \; O_H = (0, 0, 0), \; O_V = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; and \; A \in SO(3) .$$

Tensor fields $S\Theta(4)$-invariant on $\Re^3$ are physically of particular interest. A spherically symmetric space-time metric is in fact an $S\Theta(4)$-*invariant metric*. From now on this last term will be used.

Now let,

$$g_{00}(x_0,x)(dx_0 \otimes dx_0) + \Sigma_{i=1}^3\ g_{0i}(x_0,x)(dx_0 \otimes dx_i + dx_i \otimes dx_0)$$

$$+\Sigma_{i,j=1}^3\ g_{ij}(x_0,x)(dx_i \otimes dx_j),\ \ (x_0\ \in \Re,\ \ x=(x_1,x_2,x_3)\in \Re^3)\,,$$

be a C^∞ space-time metric, written as a tensor field on $\Re \times \Re^3$. A simple computation shows that it is $S\Theta(4)$-invariant. if and only if, for each $x_0 \in \Re$,

a) $g_{00}(x_0,x)$ is an $SO(3)$-invariant function, namely $g_{00}=a_{00}(x_0,||x||)$.

b) $\Sigma_1^3\ g_{0i}(x_0,x)dx_i$ is an $SO(3)$-invariant form, namely $\Sigma_1^3\ g_{0i}(x_0,x)dx_i = a_{0i}(x_0,||x||)\Sigma_1^3\ x_i dx_i$ according to Proposition 4.1.

c) $\Sigma_{i,j=1}^3\ g_{ij}(x_0,x)(dx_i \otimes dx_j)$ is an $SO(3)$-invariant tensor field, namely $\Sigma_{i,j=1}^3\ g_{ij}(x_0,x)(dx_i \otimes dx_j) = -\ a_{11}(x_0,||x||)\Sigma_1^3(dx_i \otimes dx_i) - a_{22}(x_0,||x||)$ $((\Sigma_i^3\ x_i dx_i) \otimes (\Sigma_1^3\ x_j dx_j))$ according to proposition 4.2. (The sign - is chosen for convenience of computation).

Returning to the classical notations, we see that every $S\Theta(4)$-invariant space-time metric on $\Re \times \Re^3$ can be written as

$$ds^2 = a_{00}(x_0,||x||)dx_0^2 +$$

$$2a_{01}(x_0,||x||)(xdx)dx_0 - a_{11}(x_0,||x||)dx^2 - a_{22}(x_0,||x||)(xdx)^2\,. \quad (4.1)$$

Of course, the functions $a_{00}(x_0,u), a_{01}(x_0,u), a_{22}(x_0,u)$ are supposed C^∞ on $\Re \times [0,+\infty[$, but, since $||x||$ is not differentiable at the origin, some local conditions must be satisfied, in order that the metric tensor be, too, C^∞ at $x=(0,0,0)$.

5 $S\Theta(4)$-invariant gravitational fields and Birkhoff's Theorem

The problem of defining the gravitational field of a spherical mass is clearly formulated in Newtonian mechanics, but not in the classical interpretation of general relativity.

In Newtonian mechanics, one considers, in the Euclidean space $\Re^3$, a static spherical mass the density of which, denoted by $\epsilon(\rho)$, depends only on the distance ρ from its center. Let k be the gravitational constant and denote α the radius of the spherical mass. Then, if we set,

$$m = 4\pi \int_0^\alpha \rho^2 \epsilon(\rho) d\rho$$

the Newtonian potential is given, for every $\rho \geq \alpha$, by the expression:

$$-\frac{km}{\rho}\,.$$

This last keeps its mathematical validity for any $\rho > 0$, if we suppose that $\alpha \to 0$ and $\epsilon \to +\infty$ in such a way that m does not change. Thus we arrive at the concept of mass point or punctual source, which, although not conceivable physically, is admitted in the computations ocurring in classical mechanics and quantum mechanics as well. So the concept of punctual source is based upon the fact that the expression of the potential allows to bring forward the operation $\alpha \to 0$.

In general relativity, we have also to do with the exterior gravitational field of a spherical, in general non-stationary, distribution of matter centered at $(0,0,0) \in \Re^3$. The sphere bounding the matter is then a non-Euclidean object, which is thus characterized not only by its radius $||x(t)|| = \sigma(t)$, but also by its curvature radius $\zeta(t)$. So the C^∞ functions of time $\sigma(t)$ and $\zeta(t)$ constitute the boundary conditions at finite distance. Apart from them, we have to take into account the total mass as well as the boundary conditions at infinity according to which the space-time metric (4.1) tends uniformly to a pseudo-Euclidean stationary form as $||x|| \to +\infty$. Then our problem consists in determining upon the closed set $\Re \times \left\{x \in \Re^3 : ||x|| \geq \sigma(t)\right\}$ the metric (4.1) satisfying the Einstein equations and the above boundary conditions as well. Regarding the possibility of extending the concept of punctual source to the present context, we know nothing in advance. In other words, we do not know in advance whether the solution of the problem remains valid for all $||x|| > 0$ when $\sigma(t) > 0$ is reduced to zero by a C^∞ homotopy: $H \;:\; [0,1] \times \Re \to [0,+\infty[$ such that $H(u,t) > 0$ on $[0,1[\times\Re$, $H(0,t) = \sigma(t)$ and $H(1,t) = 0$. The question is to be settled only when the solution is available. (Indeed, contrary to the Newtonian case, in the present case the answer is no).

The calssical approach to the problem is quite different. Following Birkhoff's reasoning, we can analyse it as follows:

a) The manifold $\Re \times \Re^3$ is not taken into account and the metric is considered from the outset on the manifold with boundary $\Re \times [0,+\infty[\times S^2$:

$$ds^2 = C(u,\rho)du^2 - D(u,\rho)d\rho^2 - 2E(u,\rho)dud\rho - F(u,\rho)d\omega^2 \quad (5.1)$$

Of course, (5.1) is assumed everywhere differentiable, but, as is already known, without specific conditions, there corresponds to it a space-time form on $\Re \times \Re^3$ discontinuous at $x = (0,0,0)$.

b) It is believed that there exists an implicit transformation (implicit diffeomorphism): $u = f_1(t,r)$, $\rho = f_2(t,r)$ reducing (5.1) to

$$ds^2 = (a(t,r))^2 dt^2 - ((\beta(t,r))^2 dr^2 + r^2 d\omega^2) \quad (5.2)$$

In other words, it is asserted that, whatever the components $C(u,\rho)$, $D(u,\rho)$, $E(u,\rho)$, $F(u,\rho)$ may be, there exists two differentiable functions f_1 and f_2 of

(r,t) defined globally on $\Re \times [0,+\infty[$ such that

$$C(f_1,f_2)\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial r} - D(f_1,f_2)\frac{\partial f_2}{\partial t}\frac{\partial f_2}{\partial r} - E(f_1,f_2)\left(\frac{\partial f_1}{\partial t}\frac{\partial f_2}{\partial r} + \frac{\partial f_1}{\partial r}\frac{\partial f_2}{\partial t}\right) = 0\,,$$

$F(f_1,f_2) = r^2$ and $\frac{D(f_1,f_2)}{D(r,t)} \neq 0$. This assertion is obviously erroneous. Moreover (5.2) arises froma uniquely defined space-time form on $\Re \times \Re^3$:

$$(a(t,||x||))^2 dt^2 - ((\beta(t,||x||))^2 - 1)\frac{(xdx)^2}{||x||^2} - dx^2$$

which, apart from the special case where $\beta(t,0) = 1$, is discontinuous at $x = (0,0,0)$.

c) The parameter r occuring in (5.2) is extemely misleading. It is considered either as a radial coordinate or as a true distance, but t is neither of them. In fact, $2\pi r$ is the length of circumference of a non-Euclidean circle the radius of which is not known. It seems that the parameter r was introduced for the first time by Levi-Civita, according to a reference given by Levi-Civita himself [3]. so it is convenient to call it *Levi-Civita's parameter*. The Levi-Civita parameter has nothing to do with coordinates and destroys the boundary conditions of the problem: With respect to (5.2), the spherical source has neither center nor radius, it is inexistent for the metric. Moreover, because of the Levi-Civita parameter, the metric (5.2) is inconsistent with the non-stationary exterior field. In fact, this last is generated by the radial motion of the sphere bounding the matter, namely by the motiondescribed mathematically by the boundary conditions $\sigma(t)$ and $\zeta(t)$, which are indefinable relative to (5.2). The metric (5.2) cannot describe a non-sationary field. It follows that the equations of gravitation (unless they are wrong) must give as solution a static metric (5.2). Now Birkhoff's assertion is based upon two statements:

First: The metrics (5.1) and (5.2) are equivalent.

Second: The metric (5.2) gives rise to a static solution.

Regarding the first, the equivalence is understood mathematically as well as physically. But already the mathematical (formal) equivalence is refuted by the point b).

Regarding the second, we have just ascertained that it is simply a hidden vicious circle.

d) As is expected, the solution of the Einstein equations related to (5.2) is static:

$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2\mu}{r}} - r^2 d\omega^2,\ \mu = \frac{km}{c^2}\,.$$

In fact it is the Droste solution, or, more precisely, the Droste-Hilbert solution, wrongly called Schwarzschild's solution in the literature. We have already seen that the implicit diffeomorphism considered in the point b) is in general actually inexistent. Now the discontinuity of the Droste solution ar $r = 2\mu$ proves that the implicit diffeomorphism in question is also inconsistent with the differentiable solutions of the Einstein equations. So Birkhoff's reasoning includes a deep contradiction: It presupposes a differentiability refuted by its own conclusions. Regarding the discontinuity of the solution at $r=2\mu$, it has given rise to an endless discussion without a clear premise. However, the so called "horizon $r=2\mu$" does not necessarily result from the einstein equations, but from the inadmissible implicit transformation which replaces a fundamental function, that is the function $(F(u,\rho))^{\frac{1}{2}}$ representing the curvature radius of the spheres centered at the origin, by the Levi Civita parameter.

In conclusion, the Birkhoff theorem is a pseudo-theorem which must be rejected together with the Schwarzschild and Droste solutions, in order to reexamine from the outset the problems regarding the $S\Theta(4)$-invariant gravitational fields.

In fact, the Einstein equations have non-stationary (dynamical) solutions without singularities describing the gravitational field outside a spherical mass effecting radial motion. As is shown in a previous paper [7], these solutions depend on two fundamental notions, namely the *gravitational disturbance* and *the propagation function.*

Since the external gravitational field is the extension of the internal one through the sphere $||x||=\sigma(t)$, the gravitational radiation depends on the derivatives $\sigma'(t)$ and $\zeta'(t)$, so that we may think of their pair as *the gravitational disturbance* inducing the dynamical states of the field outside the mass. Consequently, in order to determine the non-stationary solutions, we have first to clarify the propagation process of the gravitational disturbance, and so we are led, in particular, to introduce a conveniently defined *propagation function.* The relevant problems lie beyond the scope of the present short account.

We have already noticed that the vacuum solutions related to a spherical mass are inconsistent with the notion of punctual source. Of course, the validity of this statement deos not depend on the state of the field. However, its verification is easier when the field is stationary or static.

If the metric (4.1) is static, then the unknown functions occurring in it depend only on $||x||=\rho$. Moreover $a_{01}(\rho)=0$. On the other hand, since the metric is everywhere Lorentzian, we have the conditions:

$$a_{00}(\rho)>0,\ a_{11}(\rho)>0,\ a_{11}(\rho)+\rho^2 a_{22}(\rho)>0$$

for all $\rho\geq 0$, so that we can define the real functions:

$$f(\rho)=(a_{00}(\rho))^{\frac{1}{2}},\ l(\rho)=\left[a_{11}(\rho)+\rho^2 a_{22}(\rho)\right]^{\frac{1}{2}},\ g(\rho)=\rho(a_{11}(\rho))^{\frac{1}{2}},$$

which possess a clear physical and geometrical meaning. In particular $g(\rho)$ is the curvature radius of the spheres centered at the origin. Then an easy computation reduces the vacuum field equations to the following two:

$$\left(\frac{dg(\rho)}{d\rho}\right)^2=(l(\rho))^2\left(1-\frac{2\mu}{g(\rho)}\right),\ \left(\mu=\frac{km}{c^2}\right)$$

$$f(\rho)l(\rho)=c\frac{dg(\rho)}{d\rho},$$

the first of which implies $g(\rho) \geq 2\mu$. So the vacuum solution imposes on the curvature radius $g(\rho)$ the greatest lower bound 2μ, and since $g(0)=0$ and $g(\rho)>0$ for $\rho>0$, it follows that the radius, denoted by ρ_1, of the sphere bounding the matter is necessarily positive. In other words the solution is incompatible with the value $\rho_1=0$. The source is necessarily an extended body [8].

Note that $g(\rho)$ cannot take the value 2μ outside the source. In other words, we have $g(\rho)>2\mu$ at every point for which $||x||>\rho \geq \rho_1$. In fact, if $g(\rho_0)=2\mu$ for some value $\rho_0 \geq \rho$, then $\rho=\rho_1$ (since $g(\rho)$ is increasing) and the above equations give $f(\rho_0)l(\rho_0)=0$ that is either $f(\rho_0)=0$ or $l(\rho_0)=0$, which implies the degeneracy of the metric. However a degenerate space-time metric has no physical meaning. It follows that the value of ρ_0 is irrelevant physically.

5 References

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