MECHANICS. - ON THE ANALYTIC EXPRESSION THAT MUST BE GIVEN TO THE GRAVITATIONAL TENSOR IN EINSTEIN'S THEORY †

Note by the Fellow

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Foreword. While most textbooks of general relativity and research articles discuss at length the relative merits of the pseudo tensors proposed by Einstein and by other authors for representing the energy of the gravitational field, Levi Civita's definition of a true gravitational energy tensor has succumbed to Einstein's authority and is nearly forgotten. It seems however worthy of a careful re-examination, due to its unquestionable logical soundness and to the unique manner of propagation for gravitational energy that it entails.

In the present Note, after having recalled, for the reader's convenience, the leading idea and the mathematical framework of general relativity, I show how some identities (involving the derivatives of the Riemann symbols) discovered by Bianchi offer a sure criterion for introducing the so called gravitational tensor. From the analytic standpoint one has to do with a double symmetric system A_{ik} $(i, k = 0, 1, 2, 3)$, whose ten elements completely define the gravitational contribution to the local mechanical behaviour. In fact they determine the stresses as well as the energy flow and the energy density (of gravitational origin). The mechanical meaning of the system requires an analytic structure endowed with convenient invariance properties with respect to co-ordinate transformations. Such is actually the (covariant) form of the A_{ik} yielded by the above mentioned criterion. Furthermore this form provides a very expressive extension of d'Alembert's principle.

The idea of a gravitational tensor belongs to the great construction by Einstein. However its definition as given by the Author cannot be considered final. First of all, from the mathematical standpoint, it lacks the invariant character that it should instead necessarily enjoy according to the spirit of general relativity. Even worse is the fact, perceived with keen intuition by Einstein himself ([1]), that from such a definition it follows a clearly unacceptable consequence about the gravitational waves. For this point he however finds a way out in quantum theory.

The solution is however less remote: everything depends on the incorrect form assumed for the gravitational tensor. We shall see that with our determination any possibility for paradox automatically disappears.

1. GENERALITIES. - In ordinary mechanics the physical space is taken to be strictly Euclidean, and the analytical representation of phenomena is, let us say, subordinated to the (ternary) quadratic differential form dl^2 that expresses the square of the line element.

In the restricted theory of relativity one persists in considering space as Euclidean; however the equations of mechanics are no longer invariant with respect to the form dl^2; they are invariant

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([1]) *Näherungweise Integration der Feldgleichungen der Gravitation,* Sitzungsberichte der Kgl. Preussischen Ak. der Wiss., 1916, p. 696.

with respect to a quaternary form ds^2 that implies also the time t. Notoriously it has the expression

$$ds^2 = c^2 dt^2 - dl^2 \tag{1}$$

(c universal constant to be interpreted as velocity of light *in vacuo*).

It is clear that, with reference to Cartesian co-ordinates, one has again

$$dl^2 = dx^2 + dy^2 + dz^2.$$

In general relativity - new and more comprehensive conception of the natural laws, again due to Einstein - space and time do not provide a plain localisation, inert and immutable, of phenomena. They are instead affected by the latter and they react in such a way as to change the nature of ds^2.

Instead of (1), one has the fundamental form

$$ds^2 = \sum_0^3{}_{ik}\ g_{ik} dx_i dx_k, \tag{2}$$

which, with appropriate choice of the parameters x_0, x_1, x_2, x_3, reduces itself exactly to the form (1) in the limit case, when any physically perceptible action (either presence or motion of matter, of electricity, more generally of some form of energy) is lacking. As a rule, although quantitatively very close to the type (1), (2) must be considered as not given *a priori*, but intrinsically definable according to the factual circumstances. Among these it obviously appears also the universal gravitation, which, according to Einstein, deserves the privilege of depending exclusively on the coefficients g_{ik} (and on their derivatives).

The equations of the new mechanics as a whole are invariant with respect to that well determined form (2) that pertains to the specific case. In this new mechanics the theory for a given class of phenomena necessarily entails, together with relations that have their counterpart in the previous formulations (classical and relativistic of the first manner), further relations whose scope is the determination of ds^2. These are Einstein's gravitational equations (in number of 10 like the coefficients g_{ik}), that we shall consider explicitly in § 6.

2. Energy tensor. - The mechanics of continuous systems - also according to the ordinary scheme - leads to conclude that a mechanical phenomenon (occurring in a given range of values for x, y, z, t) is well known when the following elements: stresses, momentum, energy flow and energy density, are assigned (as functions of position and of time).

In relativistic mechanics the vector $\mathbf{q}$, that represents the momentum density, is linked to the energy flow χ by the relation

$$\mathbf{q} = \frac{1}{c^2}\chi.$$

It is convenient to avail of the single vector

$$-\mathbf{f} = c\mathbf{q} = \frac{1}{c}\chi, \tag{3}$$

that can be considered as the energy flow occurring in a light-second (time interval during which light travels the length unit). Let us notice, to avoid any misunderstanding, that we do not mean to fix in this way the time unit: it remains generic, like the other two fundamental units.

Let us write for short

(4) $$y_0 = ct, \quad y_1 = x, \quad y_2 = y, \quad y_3 = z.$$

With reference to these variables, let us introduce a symmetric tensor T_{ik} defined in this way: for $i, k = 1, 2, 3$, T_{ik} is the component along the y_k axis of the specific stress exerted on a surface element normal to the y_i axis ([2]) (or vice versa, by exchanging i and k); $T_{i0} = T_{0i}$ is identified with the component f_i of the vector $\mathbf{f}$; T_{00} is finally the energy density.

3. Reference to arbitrary co-ordinates. - If four arbitrary (independent) combinations x_0, x_1, x_2, x_3 are substituted for the y_i, the form

(1′) $$ds^2 = c^2 dy_0^2 - (dy_1^2 + dy_2^2 + dy_3^2)$$

takes the general expression (2). However the qualitative restrictions

(5) $$g_{00} > 0, \quad g_{ii} < 0 \qquad (i = 1, 2, 3),$$

shall hold whenever the parameter x_0 (individually varied) is apt to reflect the intuitive notion of time, while the remaining x_i can in some way be interpreted as actual space co-ordinates.

When this holds, *we shall interpret as energy tensor in arbitrary co-ordinates x_i that double covariant system T_{ik} which, when referred to the y, is specified in the way shown above.*

From the very covariance formulae that define the elements of the system T_{ik} one derives, for each of them, an interpretation in general co-ordinates x. And precisely one finds ([3]) that:

$$\frac{T_{ik}}{\sqrt{g_{ii}g_{kk}}} = \frac{T_{ki}}{\sqrt{g_{ii}g_{kk}}} \qquad (i, k = 1, 2, 3)$$

represents the (orthogonal) component along the line x_i (the one along which only x_i varies) of the stress exerted on a surface element normal to the line x_k (or vice versa, by exchanging i and k);

$$\frac{T_{i0}}{\sqrt{-g_{00}g_{ii}}} = \frac{T_{0i}}{\sqrt{-g_{00}g_{ii}}} \qquad (i = 1, 2, 3)$$

represents the component of $\mathbf{f}$ along the line x_i (when one imagines decomposing the vector $\mathbf{f}$ with respect to the trihedron of the co-ordinate lines); finally

$$\frac{T_{00}}{g_{00}}$$

is the density of the energy distribution in the space $(x_1,\ x_2,\ x_3)$, [to which the metric determination corresponding to $-ds^2$ for $dx_0 = 0$ is attributed].

4. Linear invariant and divergence of the energy tensor. - We adhere to the usual notation of the absolute differential calculus. Therefore we represent with $g^{(ik)}$ the elements reciprocal to the coefficients g_{ik}, and with T_{ikl} $(i, k, l = 0, 1, 2, 3)$ the system covariantly derived from T_{ik} according to the fundamental form. At present, as already in the previous §, we shall

([2]) with the convention (usual in hydrodynamics) that a positive normal stress corresponds to pressure.

([3]) Dwelling here on the way of deduction would be out of place. However I allow myself to notice that there is no need of actual calculations: one can avail of an appropriate adjustment of the methods of the absolute differential calculus to the indefinite ds^2 that encompass space and time.

assume this form to be (1') which, when referred to arbitrary co-ordinates x, takes the generic expression (2).

By setting

$$T = \sum_0^3{}_{ik}\, g^{(ik)} T_{ik}, \tag{6}$$

one defines an invariant, that is just called *linear invariant* or *scalar of the energy tensor*.

One calls instead *divergence* of the same energy tensor the simple covariant system (or four-dimensional vector)

$$F_i = \sum_0^3{}_{kl}\, g^{(kl)} T_{ikl} \quad (i = 0, 1, 2, 3). \tag{7}$$

The mechanical meaning of the divergence (like the meaning of T, which I neglect to notice, because it is immediate) becomes clear when one goes back to the variables y. With respect to such variables $g^{(ik)} = 0$ $(i \neq k)$, $g^{(00)} = 1$, $g^{(ii)} = -1$ $(i = 1, 2, 3)$, and the covariant differentiation coincides with the ordinary one.

Then one has

$$F_i = \frac{\partial T_{i0}}{\partial y_0} - \sum_1^3{}_k \frac{\partial T_{ik}}{\partial y_k} \quad (i = 1, 2, 3),$$

$$F_0 = \frac{\partial T_{00}}{\partial y_0} - \sum_1^3{}_k \frac{\partial T_{0k}}{\partial y_k}.$$

Let us remind of § 2 and notice that, due to (3), the components T_{i0} are identical with $-cq_i$ (q_i components of the momentum density $\mathbf{q}$). Therefore it obviously appears from the first three equations written above (when ct is substituted for y_0) that $-F_i$ $(i = 1, 2, 3)$ are components of the external force $\mathbf{F}$ applied to the system (per unit volume). The last equation, when [always according to (3)] the T_{0k} are taken in the form $-\frac{1}{c}\chi_k$ (χ_k components of the energy flow χ), finally shows that cF_0 is the power density, *i.e.* the energy given from outside to the system per time and volume units. One can also say, if desired, that F_0 represents the energy given to the system per volume unit in a light-second. Hence it turns out in particular that for an isolated system the divergence is vanishing.

When one avails as reference of arbitrary co-ordinates x_i, the covariant character of the simple system F_i immediately allows one to interpret

$$\frac{-F_1}{\sqrt{-g_{11}}}, \quad \frac{-F_2}{\sqrt{-g_{22}}}, \quad \frac{-F_3}{\sqrt{-g_{33}}}$$

as components of $\mathbf{F}$ along the co-ordinate lines x_1, x_2, x_3;

$$\frac{F_0}{\sqrt{g_{00}}}$$

as energy given in one light-second to the unit volume of the system.

5. Transition to general relativity. - Despite the reference to general co-ordinates, up to now we have supposed to deal with a Euclidean ds^2. Formally the situation happens to be the same also with a ds^2 essentially not reducible to the type (1'), if however:

a) x_0 can be interpreted as time and the other three co-ordinates as space parameters since, in keeping with this, the inequalities (5) hold true;

b) the usual mechanical intuitions are preserved (at an infinitesimal scale), hence it is possible to attribute a definite meaning to local measurements of force, stresses, energy flow and energy density. In such conditions the energy tensor is uniquely defined through the ratios

$$\frac{T_{ki}}{\sqrt{g_{ii}g_{kk}}} \quad (i,k=1,2,3)\ ; \quad \frac{T_{0i}}{\sqrt{-g_{00}g_{ii}}} \quad (i=1,2,3)\ ; \quad \frac{T_{00}}{g_{00}},$$

given in § 3.

We shall hereafter assume our ds^2 to be *a priori* arbitrary (apart from the restrictions given above); naturally one shall take this ds^2 as fundamental form.

6. The equations of the gravitational field. - Let $g_{ij,hk}$ $(i,j,h,k=0,1,2,3)$ indicate the Riemann symbols of the first kind belonging to a generic quaternary ds^2 like (2). Due to their covariance, the positions

$$G_{ik} = \sum_0^3{}_{jh}\ g^{(jh)} g_{ij,hk} \quad (i,k=0,1,2,3) \tag{8}$$

define a double covariant system.

Let us remind of the formulae ([4])

$$g_{ij,hk} = \sum_0^3{}_{\nu}\ g_{j\nu}\{i\nu,hk\},$$

$$\{i\nu,hk\} = \frac{\partial}{\partial x_k}\left\{ {i\ h \atop \nu} \right\} - \frac{\partial}{\partial x_h}\left\{ {i\ k \atop \nu} \right\} + \sum_0^3{}_{l}\ \left[\left\{ {i\ h \atop l} \right\}\left\{ {l\ k \atop \nu} \right\} - \left\{ {i\ k \atop l} \right\}\left\{ {l\ h \atop \nu} \right\}\right],$$

that connect the Riemannian symbols of the first kind with those of the second kind, and the latter with the Christoffel symbols (again of the second kind). One immediately recognises that (8) are equivalent to

$$\begin{aligned} G_{ik} &= \sum_0^3{}_{h}\ \{ih,hk\} = \\ &= \sum_0^3{}_{h}\ \left[\frac{\partial}{\partial x_k}\left\{ {i\ h \atop h} \right\} - \frac{\partial}{\partial x_h}\left\{ {i\ k \atop h} \right\}\right] + \sum_0^3{}_{hl}\ \left[\left\{ {i\ h \atop l} \right\}\left\{ {k\ l \atop h} \right\} - \left\{ {i\ k \atop l} \right\}\left\{ {l\ h \atop h} \right\}\right]. \end{aligned} \tag{8'}$$

The linear invariant of the double system G_{ik}

$$G = \sum_0^3{}_{ik}\ g^{(ik)} G_{ik} \tag{9}$$

([4]) Bianchi, *Lezioni di geometria differenziale*, vol. I [Pisa, Spoerri, 1902], p. 72.

will be called *average curvature* of our ds^2 ([5]). With these positions, Einstein's gravitational equations are:

$$G_{ik} - \frac{1}{2} g_{ik} G = -\kappa T_{ik}, \tag{10}$$

where κ depends on the constant f of universal gravitation and on c according to the formula

$$\kappa = \frac{8\pi f}{c^4}. \tag{11}$$

I remark in passing that the homogeneity of both sides of (10) can be checked if one imagines referring to mutually homogeneous parameters x_0, x_1, x_2, x_3, *e.g.* lengths, like (for the Euclidean ds^2) the y defined by the positions (4). The coefficients g_{ik} are then pure numbers, and the left-hand sides have clearly the dimensions l^{-2}. On the other hand all the T_{ik} (specific stresses apart from numerical factors, etc.) have in this case the same dimensions, and precisely $ml^{-1}t^{-2}$. One has further

$$[f] = m^{-1} l^3 t^{-2} \quad , \quad [\kappa] = m^{-1} l^{-1} t^2,$$

hence also the right-hand sides have actually the dimensions l^{-2}.

7. Formal validation derived from the Bianchi identities. - The covariant derivatives of the Riemann symbols are linked by very remarkable relations due to Bianchi ([6]), that can be resumed in the formula

$$g_{ij,hkl} + g_{jl,hki} + g_{li,hkj} = 0 \quad (i, j, h, k, l = 0, 1, 2, 3),$$

or, due to well known properties of the Riemann symbols, in the equivalent formula

$$g_{ij,hkl} + g_{il,khj} - g_{lj,hki} = 0.$$

Let us multiply by $\frac{1}{2} g^{(kl)} g^{(jh)}$ and sum with respect to k, l, j, h; when the sum is accomplished, let us exchange in the second term j with l and h with k. The second term thus becomes identical with the first, and one gets

$$\sum_0^3 kljh \; g^{(kl)} g^{(jh)} g_{ij,hkl} - \frac{1}{2} \sum_0^3 kljh \; g^{(kl)} g^{(jh)} g_{lj,hki} = 0 \quad (i = 0, 1, 2, 3).$$

On the other hand, the covariant differentiation of (8) - by reminding of Ricci's lemma, according to which the coefficients of the fundamental form have vanishing covariant derivative - yields

$$G_{ikl} = \sum_0^3 jh \; g^{(jh)} g_{ij,hkl}.$$

From the expression (9) of G, that can be written as

([5]) This name is obviously derived from the geometric meaning that G would take, if ds^2 would be positive definite.

([6]) see loc. cit., p. 351.

$$G = \sum_0^3 {}^{kl}\ g^{(kl)} G_{lk},$$

by covariant differentiation one gets

$$\frac{\partial G}{\partial x_i} = G_i = \sum_0^3 {}^{kl}\ g^{(kl)} G_{lki} = \sum_0^3 {}^{kljh}\ g^{(kl)} g^{(jh)} g_{lj,hki}.$$

The resulting combinations of the Bianchi identities thus become

$$\sum_0^3 {}^{kl}\ g^{(kl)} G_{ikl} - \frac{1}{2} G_i = 0 \quad (i = 0, 1, 2, 3). \tag{12}$$

They contain the validation of the gravitational equations (10) from the mathematical standpoint. Here is why: the right-hand sides of (10) constitute a double system with vanishing divergence ([7]). If one requires the system (10) to be complete [*i.e.* no condition is imposed on ds^2 beyond the exterior circumstances resumed in T_{ik}], also the divergence of the first-hand sides, hence of the system

$$G_{ik} - \frac{1}{2} g_{ik} G,$$

must identically vanish. This fact is just expressed by the equations (12).

8. Gravitational (or inertial) tensor. - Generalisation of d'Alembert's principle. - If we set for short

$$A_{ik} = \frac{1}{\kappa} \left\{ G_{ik} - \frac{1}{2} g_{ik} G \right\}, \tag{13}$$

the gravitational equations (10) read

$$T_{ik} + A_{ik} = 0 \quad (i, k = 0, 1, 2, 3). \tag{10'}$$

In them we interpret A_{ik} as components of an energy tensor due to the space-time environment, *i.e.* exclusively dependent on the coefficients of ds^2. Such a tensor can be equally well named either *gravitational* or *inertial* ([8]) because both gravitation and inertia depend on ds^2. Therefore (10') give rise to the following proposition:

The nature of ds^2 is always such as to balance all mechanical actions; in fact the sum of the energy tensor and of the inertial one identically vanishes.

([7]) In fact T_{ik} include the contribution of all the phenomena that occur at the considered place and time (apart from gravitation). One deals anyway with an isolated system in the ordinary sense of the word. Therefore force and power must vanish within each of its elementary portions.

([8]) See for instance W. De Sitter, *On the relativity of rotation in Einstein's theory*, Proc. of K. Ak. van Wet. te Amsterdam, vol. XIX, 1916, p. 530 (in footnote).

One is naturally led to associate this proposition with d'Alembert's principle "the lost forces (*i.e.* directly applied forces and inertial ones) balance each other". The equilibrium expressed by (10') is just the most complete occurrence that can be conceived from the mechanical standpoint. In fact, not only the total force applied to each single element comes to vanish, but also stresses, energy flow and energy density (by taking inertia into account through A_{ik}) behave in this way.

It is clear that this total lack of mechanical entities pertains to isolated systems. Let us introduce in the field of such a system for instance a bit of matter (and for simplicity the ensuing alteration of the field is supposed negligible); several external actions coming from the system are exerted on the extra matter. In the ideal case of the mass point, these can be summarised in a law of motion (geodesic with respect to the four-dimensional ds^2). It contains in particular the ordinary dynamics of a point subjected to conservative forces.

It must be remarked that Einstein's fundamental equations, connected here with d'Alembert's principle, have been already derived, by Einstein himself and, in a more complete way, by Lorentz and by Hilbert ([9]), from the appropriate variation of a unique integral. In this way also Hamilton's principle is extended to the new mechanics.

9. Einstein's misunderstanding about the gravitational tensor. - I remember, although it may be superfluous, that by setting

$$A_i^{(j)} = \sum_0^3 k\, g^{(jk)} A_{ik} \quad (i,j=0,1,2,3),$$

from any double covariant system A_{ik} one immediately gets a mixed system $A_i^{(j)}$ (covariant with respect to the index i and contravariant with respect to the index j). It follows that, in order to identify the gravitational tensor with respect to certain variables, it makes no difference if one fixes either the elements A_{ik} or their linear combinations $A_i^{(j)}$. With this proviso let us come to the explicit expressions proposed by Einstein ([10]) for the $A_i^{(j)}$, and by him called $\sqrt{-g}\; t_i^j$ (g is the discriminant of ds^2).

They are

$$\sqrt{-g}\; t_i^j = \frac{1}{2}\left\{ G^* \varepsilon_i^j - {\sum_0^3}' hk\, \frac{\partial G^*}{\partial g_j^{(hk)}} g_i^{(hk)} \right\} \quad (i,j=0,1,2,3). \tag{14}$$

Here ε_i^j is as usual either zero or one according to whether the indices are different or equal; $g_j^{(hk)}$ stands for $\partial g^{(hk)}/\partial x_j$; the summation ${\sum_0^3}' hk$ must be extended to all the combinations with repetition of the indices h and k; finally the function

$$G^* = -\sum_0^3 ik\, g^{(ik)} \sum_0^3 hl \left[\left\{ {i\ h \atop l} \right\} \left\{ {k\ l \atop h} \right\} - \left\{ {i\ k \atop l} \right\} \left\{ {l\ h \atop h} \right\} \right]. \tag{15}$$

([9]) See for instance pp. 707-709 of De Sitter's report, *On Einstein's theory of gravitation...*, Monthly Notices, vol. LXXVI, 1916.

([10]) Firstly with reference to special variables, then by extending their validity; eventually by attributing to them a general character. See in particular the recent Note: *Hamiltonsches Prinzip und allgemeine Relativitätstheorie,* Sitzungsberichte der Kgl. Preussischen Ak. der Wiss., 1916, pp. 1111-1116.

must be understood (as it is obviously allowed) as reduced to depend only on the arguments $g^{(hk)}$, $g_j^{(hk)}$ before being subjected to partial differentiation with respect to the latter.

The inappropriateness of the positions (14) from the mathematical standpoint is easily acknowledged. It is sufficient for instance to derive from them the expression that should be assumed by the linear invariant, *i.e.*

$$\sqrt{-g}\sum_0^3{}_i\; t_i^i = \frac{1}{2}\left\{4G^* - \sum_0^3{}_i\sum_0^3{}'_{hk}\; \frac{\partial G^*}{\partial g_i^{(hk)}} g_i^{(hk)}\right\}.$$

Since G^*, according to (15), is quadratic and homogeneous with respect to the Christoffel symbols, hence also with respect to $g_i^{(hk)}$, by virtue of Euler's theorem

$$\sum_0^3{}_i\sum_0^3{}'_{hk}\; \frac{\partial G^*}{\partial g_i^{(hk)}} g_i^{(hk)} = 2G^*,$$

and *the invariant in question should reduce itself to* G^*.

Now it is well known ([11]) that differential invariants of the 1^0 order which are intrinsic *i.e.*, like G^*, exclusively formed with the coefficients of ds^2 and with their first derivatives, do *not* exist. This is enough to render, at least in general, not admissible the form of the gravitational tensor taken by Einstein. The latter however had already felt some uneasiness, in particular when ([12]), after having outlined with genial simplicity the theory of the gravitational waves, he was led to the unacceptable result that also *spontaneous* waves should as a rule give rise to dispersion of energy through irradiation.

"Since this fact" - these are his words - "should not happen in nature, it seems likely that quantum theory should intervene by modifying not only Maxwell's electrodynamics, but also the new theory of gravitation".

Actually there is no need of reaching to quanta. It is enough to correct the formal expression of the gravitational tensor in the way shown here. Then the possibility of being confronted with consequences not corresponding to the physical intuition is *a priori* excluded, in the case either of free waves or of another *purely* gravitational phenomenon. In fact, by virtue of (10') or, if one likes, of the generalised d'Alembert's principle, when the energy tensor T_{ik} vanishes, the same occurrence must happen to the gravitational tensor A_{ik}. This fact entails total lack of stresses, of energy flow, and also of a simple localisation of energy.

([11]) See for instance Ricci et Levi-Civita, *Méthodes de calcul différentiel absolu et leurs applications,* Matematische Annalen, B. 54, 1900, p. 162.

([12]) In the Note already cited at the beginning.