Physics. - '*The field of a single centre in* EINSTEIN'S *theory of gravitation, and the motion of a particle in that field.*'.
By J. Droste. (Communicated by Prof. H. A. Lorentz).

(Communicated in the meeting of May 27, 1916).

In two communications[1] I explained a way for the calculation of the field of one as well as of two centres at rest, with a degree of approximation that is required to account for all observable phenomena of motion in these fields. For this I took as a starting-point the equations communicated by EINSTEIN in 1913[2] EINSTEIN has now succeeded in forming equations which are covariant for all possible transformations[3] and by which the motion of the perihelion of Mercury is entirely explained[4]. The calculation of the field should henceforth be made from the new equations; we will make a beginning by calculating the field completely and not, as before, only the terms of the first and second order. After this, we investigate the motion of a body, so small that it does not produce any observable change in the original field.

1. The equations for the calculation of the field can be got from a principle of variation. Where matter is absent ($T_{ij} = 0$) the variation of the integral

$$\int\int\int\int G\sqrt{-g}dx_1dx_2dx_3dx_4$$

must be zero, if the variations of all g's and their first derivatives be zero at the three-dimensional limits of the four-dimensional region over which the integral is extended. Here G represents the quantity

$$G = 2\Sigma_{ij}g^{ii}\left(\frac{\partial}{\partial x_i}\left\{ {ij \atop j} \right\} - \frac{\partial}{\partial x_j}\left\{ {ii \atop j} \right\}\right) + 2\Sigma_{ijk}\left(\left\{ {ij \atop k} \right\}\left\{ {ki \atop j} \right\} - \left\{ {ii \atop k} \right\}\left\{ {kj \atop j} \right\}\right)g^{ii}, \quad (1)$$

$$\left\{ {ij \atop k} \right\} = \Sigma_l g^{kl}\left[{ij \atop l} \right], \ \left[{ij \atop l} \right] = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right).$$

For a centre at rest and symmetrical in all directions it is easily seen that

$$ds^2 = \omega^2 dt^2 - u^2 dr^2 - v^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right), \quad (2)$$

ω, u, v only depending on r, and (ϑ, φ) representing polar coordinates.

Now, if g_{ij} and therefore also g^{ij} are all zero, if $i \neq j$, G breaks up into six pieces, each of them relating to two indices. We collect the terms belonging to α and β and name their sum $Gx_\alpha x_\beta$.

[1] Volume XVII p. 998 and vol. XVIII p. 760.

[2] 'Entwurf einer verallgemeinerten Relativitatstheorie und einer Theorie der Gravitation', TEUBNER. Or : Zeitschrift fur Mathematik und Physik, vol 62.

[3] 'Die Feldgleichungen der Gravitation' Sitzungsberichte der Kon. Preuss. Akad. der Wiss. 1915, p. 844.

[4] 'Erklarung der Perihelbewegung des Merkur aus der allgemeinen Relativitatstheorie' Sitzungsberichte der Kon. Preuss. Akad. der Wiss. 1915, p.831.

Now, if a, b, c represent three different indices,

$$\left[{ab \atop c}\right] = 0,\ \left[{aa \atop c}\right] = -\frac{1}{2}\frac{\partial g_{aa}}{\partial x_c},\ \left[{ab \atop a}\right] = \frac{1}{2}\frac{\partial g_{aa}}{\partial x_b},\ \left[{aa \atop a}\right] = \frac{1}{2}\frac{\partial g_{aa}}{\partial x_a}.$$

So

$$\left\{{ab \atop c}\right\} = 0,\ \left\{{aa \atop c}\right\} = -\frac{1}{2}g^{cc}\frac{\partial g_{aa}}{\partial x_c},\ \left\{{ab \atop a}\right\} = \frac{1}{2}g^{aa}\frac{\partial g_{aa}}{\partial x_b},\ \left\{{aa \atop a}\right\} = \frac{1}{2}\frac{\partial g_{aa}}{\partial x_a}.$$

Let the second sum in (1) contribute to $Gx_\alpha x_\beta$ the terms, in which $i = \alpha, j = \beta$, or $i = \beta, j = \alpha$. By taking for α and β successively the six couples of indices and adding the expressions, we get exactly the first sum of (1).

Let the second sum in (1) contribute to $Gx_\alpha x_\beta$ those terms in which one of the *differentiated* $g's$ contains the index α, the other β. So that the sum too will have been broken up into six pieces, one of which relates to α and β.

In that way we obtain

$$Gx_\alpha x_\beta = g^{\alpha\alpha}\frac{\partial}{\partial x_\alpha}\left(g^{\beta\beta}\frac{\partial g_{\beta\beta}}{\partial x_\alpha}\right) + g^{\alpha\alpha}\frac{\partial}{\partial x_\beta}\left(g^{\beta\beta}\frac{\partial g_{\alpha\alpha}}{\partial x_\beta}\right) + g^{\beta\beta}\frac{\partial}{\partial x_\beta}\left(g^{\alpha\alpha}\frac{\partial g_{\alpha\alpha}}{\partial x_\beta}\right) +$$
$$+ g^{\beta\beta}\frac{\partial}{\partial x_\alpha}\left(g^{\alpha\alpha}\frac{\partial g_{\beta\beta}}{\partial x_\alpha}\right) + g^{\alpha\alpha}g^{\beta\beta}\ \Sigma_{\alpha\neq i\neq\beta}\ g^{ii}\frac{\partial g_{\alpha\alpha}}{\partial x_i}\frac{\partial g_{\beta\beta}}{\partial x_i}. \tag{3}$$

The equations of the field being covariant for all transformations of the coordinates whatever, we are at liberty to choose instead of r a new variable which will be such a function of r, that in ds^2 the coefficient of the square of its differential becomes unity. That new variable we name r again and we put

$$ds^2 = \omega^2 dt^2 - rd^2 - v^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right) \tag{4}$$

ω and v only depend on r. We now find

$$G_{tr} = -\frac{4\omega''}{\omega},\ G_{\vartheta r} = G_{\varphi r} = -\frac{4v''}{v},\ G_{t\vartheta} = G_{t\varphi} = -\frac{4v'\omega'}{v\omega},\ G_{\vartheta\varphi} = \frac{4}{v^2} - \frac{4v'^2}{v^2}.$$

In these equations accents represent differentiation with respect to r. So

$$G = \frac{4}{v^2} - \frac{4v'^2}{v^2} - \frac{8v'\omega'}{v\omega} - \frac{8v''}{v} - \frac{4\omega''}{\omega}.$$

Now, as $\sqrt{-g} = v^2\omega\sin\vartheta$, the function to be integrated in the principle of variation becomes

$$4\left(\omega - \omega v'^2 - 2vv'\omega' - 2v\omega v'' - v^2\omega''\right)\sin\vartheta.$$

We now apply the principle to the region $t_1 \leq t \leq t_2, r_1 \leq r \leq r_2$. By effecting the integrations with respect to t, ϑ and φ we find the condition

$$\delta\int_{r_1}^{r_2}\left(\omega - \omega v'^2 - 2vv'\omega' - 2v\omega v'' - v^2\omega''\right)dr = 0.$$

This gives us

$$2vv'' + v'^2 = 1 \tag{5}$$

and

$$v\omega'' + v'\omega' + \omega v'' = 0. \tag{6}$$

These are the equations of the field required.

2. To solve (6), we introduce instead of r the quantity $x = v'$ as an independent variable by which, on taking account of (5), (6) changes into

$$\left(1 - x^2\right) \frac{d^2\omega}{dx^2} - 2x\frac{d\omega}{dx} + 2\omega = 0.$$

This equation is satisfied by $\omega = x$. The other particular solution is now also easyily found, viz.

$$\omega = 1 - \frac{1}{2}x \log \frac{1-x}{1+x}.$$

But we want ω to be a finite constant if $v' = 1$ (for $r = \infty$). Then ω must be equal to x, if we take the constant to be 1 (the speed of light then approaches to 1 at large distances from the centre).

The introduction of x in (5) gives

$$\frac{dv}{dx} = \frac{2xv}{1-x^2},$$

from which we immediately find

$$v = \frac{\alpha}{1-x^2},$$

α being a constant of integration.

Differentiating this relation with respect to r, we get

$$v' = \frac{2\alpha x}{\left(1-x^2\right)^2} \frac{dx}{dr}$$

or, v' being equal to x,

$$dr = \frac{2\alpha dx}{\left(1-x^2\right)^2}.$$

So (4) changes into

$$ds^2 = x^2 dt^2 - \frac{4\alpha^2}{\left(1-x^2\right)^4} dx^2 - \frac{\alpha^2}{\left(1-x^2\right)^2} \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right).$$

So we have now been led again to introduce another variable instead of r, viz. x. The form obtained leads us to introducing the variable $\xi = 1 - x^2$. Then

$$ds^2 = \left(1-\xi\right) dt^2 - \frac{4\alpha^2}{\left(1-\xi\right)\xi^4} d\xi^2 - \frac{\alpha^2}{\xi^2} \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right).$$

Lastly we put

$$\xi = \frac{\alpha}{r}.$$

This r is not the same as occurs in (4). We obtain

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2 \left(d\theta^2 + \sin^2 \vartheta d\varphi^2\right). \qquad (7)$$

We have chosen the coordinates in a particular manner; it is now of course also very easy to introduce for r another variable, which is a function of r[5].

3. From (7) we can immediately deduce some conclusions. The point (r, ϑ, φ) lies at a distance

$$\delta = \int_\alpha^r \frac{dr}{\sqrt{1 - \frac{1}{r}}} = r\sqrt{1 - \frac{\alpha}{r}} + \alpha \log\left(\sqrt{\frac{r}{\alpha} - 1} + \sqrt{\frac{r}{\alpha}}\right), \qquad (8)$$

from the point, where the radius intersects sphere $r = \alpha$, if $r > \alpha$ and supposing that (7) remains valid up to $r = \alpha$. In future we will always make these two suppositions; as we shall see, that a moving particle outside sphere $r = \alpha$ can never pass that sphere, we may, in studying its motion, disregard the space $r < \alpha$. Should (7) cease to be valid as soon as r becomes $< R$, we need only exclude the space $r < R$ from the conclusions which will still be made, to make them valid again.

If r be very large with respect to α, the proportion $\delta : r$ approaches 1.

The circumference of a circle $r = const.$ is $2\pi r$ by (7); this shows how r can be measured. Circle α has the circumference $2\pi\alpha$.

One might in (7) perform a sunstitution $t = f(r, \tau)$. Then a term containing $drd\tau$ would arise and the velocity c of light, travelling along r, would have to be calculated from an equation of the form

$$F_1(r, \tau) + F_2(r, \tau)c - F_3(r, t)c^2 = 0$$

and would have *two* values, one for light coming from the centre, the other for light moving towards it. Moreover these values would depend on t. In consequence of the last fact we should not name the field stationary and the first fact does not agree with the way in which time is compared in two different places. So, if we want to retain both advantages, such s substitution is not allowed, though it may, of course, always be done, if we are willing to give up these advantages.

We will point out that, as (7) is known now, G can be found as a function of r. The result is $G = 0$, as it must always be found where matter is absent.

[5] After the communication to the Academy of my calculations, I discovered that also K. SCHWARZSCHILD has calculated the field. Vid.: Sitzungsberichte der der Kon. Preuss. Akad. der Wiss. 1916, page 189. Equation (7) agrees with (14) there, if R is read instead of r.

4. We now proceed to the calculation of the equations of motion of a particle in the field.

The equations of motion express the fact that the first variation of the integral

$$\int_{t_1}^{t_2} L dt$$

will be zero, if the varied positions for $t = t_1$ and $t = t_2$ are the same as the actual ones. L represents the quantity

$$L = \frac{ds}{dt} = \sqrt{1 - \frac{\alpha}{r} - \frac{\dot{r}^2}{1 - \frac{\alpha}{r}} - r^2 \dot{\vartheta}^2 - r^2 \sin^2 \vartheta \dot{\varphi}^2}, \tag{9}$$

where

$$\dot{r} = \frac{dr}{dt}, \ \dot{\vartheta} = \frac{d\vartheta}{dt}, \ \dot{\varphi} = \frac{d\varphi}{dt}.$$

One of the equations of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 0$$

or

$$\frac{r^2 \sin^2 \vartheta \ \dot{\varphi}}{L} = const.,$$

which proves that $\dot{\varphi}$, once being zero, keeps that value.

Now, as we can always choose ϑ and φ in such a way that $\dot{\varphi}$ becomes zero for a certain value of t and as φ will then always remain zero, the motion takes place in a plane.

We choose the coordinates in such a manner, that this plane becomes the plane $\vartheta = \frac{\pi}{2}$. Then (9) passes into

$$L = \sqrt{1 - \frac{\alpha}{r} - \frac{\dot{r}^2}{1 - \frac{\alpha}{r}} - r^2 \dot{\varphi}^2}. \tag{10}$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 0, \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0. \tag{11}$$

From these two it follows that

$$\frac{d}{dt}\left(L - \dot{r}\frac{\partial L}{\partial \dot{r}} - \dot{\varphi}\frac{\partial L}{\partial \dot{\varphi}}\right) = 0$$

or

$$\frac{d}{dt}\left(\frac{1 - \frac{\alpha}{r}}{L}\right) = 0. \tag{12}$$

Instead of the two equations (11) we may consider the system, consisting of (11) and (12). The two systems are equivalent only in case $\dot{r} \neq 0$; so for the circular motion we shall have to return to the second equation (10).

We now obtain

$$\frac{1-\frac{\alpha}{r}}{L} = const., \ \frac{r^2\dot{\varphi}}{L} = const.,$$

and so

$$\frac{r^2\dot{\varphi}}{1-\frac{\alpha}{r}} = const.$$

This yields the equations

$$\frac{1}{1-\frac{\alpha}{r}} - \frac{\dot{r}^2}{\left(1-\frac{\alpha}{r}\right)^3} - \frac{r^2\dot{\varphi}^2}{\left(1-\frac{\alpha}{r}\right)^2} = A \tag{13}$$

and

$$\frac{r^2\dot{\varphi}^2}{1-\frac{\alpha}{r}} = B. \tag{14}$$

We will now just express the quantities $\ddot{\varphi}$ and $\ddot{r}$ in $\dot{\varphi}$, r and $\dot{r}$; this is easily done by differentiating (13) and (14) with respect to t. The result is

$$\ddot{\varphi} = \frac{\dot{\varphi}\dot{r}}{1-\frac{1}{r}}\,\frac{\alpha}{r^2} - \frac{2\dot{\varphi}\dot{r}}{r}, \tag{15}$$

and

$$\ddot{r} = -\frac{\alpha}{2r^2}\left(1-\frac{\alpha}{r}\right) + \frac{3}{2}\frac{\alpha}{r^2}\frac{\dot{r}^2}{1-\frac{\alpha}{r}} + r\dot{\varphi}^2\left(1-\frac{\alpha}{r}\right). \tag{16}$$

5. From (15) and (16) it follows if $\dot{r} = \dot{\varphi} = 0$

$$\ddot{\varphi} = 0, \ \ddot{r} = -\frac{\alpha}{2r^2}\left(1-\frac{\alpha}{r}\right).$$

This is the acceleration in case of a particle at rest. It is directed towards the centre.

$\ddot{r}$ has its greatest value (at rest) at the distance $r = \frac{3}{2}\alpha$ from the centre; the greatest value of $\ddot{\delta}$ is attained for $r = \frac{5}{4}\alpha$.

6. *The motion may be circular.* As $\dot{r}$ is then continually zero, we return to the equations (11). the second shows

$$\frac{\partial L}{\partial r} = 0,$$

i.e.

$$\dot{\varphi}^2 = \frac{\alpha}{2r^3}. \tag{17}$$

Substituting this in (10) and putting $\dot{r} = 0$ we find

$$L^2 = 1 - \frac{3\alpha}{2r},$$

so that r must be $> \frac{3}{2}\alpha$, if L^2 or, what comes to the same thing, ds^2 shall be positive.

Formula (17) is the same as in NEWTON'S theory.

7. We will now consider the case of $\dot{\varphi}$ being continually zero, i.e. that the particle always moves on the same radius. From (13) we easily conclude (we shall afterwoods show this in general i.e. if $\dot{\varphi}$ be not identical zero) that the particle never reaches sphere $r = \alpha$.

If we call

$$\dot{\delta} = \frac{\dot{r}}{\left(1 - \frac{\alpha}{r}\right)^{\frac{1}{2}}}, \quad \ddot{\delta} = \frac{d\dot{\delta}}{dt}$$

for abbreviation velocity and acceleration, then (13) gives us for the velocity the formula

$$\dot{\delta}^2 = \left(1 - \frac{\alpha}{r}\right)\left(1 - A + A\frac{\alpha}{r}\right) \tag{18}$$

and (16) for the acceleration

$$\ddot{\delta} = -\frac{\alpha}{2r^2}\left[\sqrt{1 - \frac{\alpha}{r}} - \frac{2\dot{\delta}^2}{\sqrt{1 - \frac{\alpha}{r}}}\right]. \tag{19}$$

If we substitute (18) in (19) we obtain

$$\ddot{\delta} = \frac{\alpha}{2r^2}\left(1 - 2A + 2A\frac{\alpha}{r}\right)\sqrt{1 - \frac{\alpha}{r}}. \tag{20}$$

From (19) it follows, that the algebraic value of the acceleration only depends on the position and the velocity of the particle and does not change if we reverse the direction of the velocity. The constant A is never negative (as $L > 0$). If A lies between 0 and 1 ($A = 1$ included), then every value of r is possible according to (18). We then have a particle moving towards infinity or coming from it. For this motion the acceleration will, according to (20), once become zero, if $2A - 1 > 0$, i.e. $A > \frac{1}{2}$, viz. for

$$r = \frac{2A\alpha}{2A - 1};$$

for greater values of r the acceleration is directed towards the centre (attraction), for smaller values of r from the centre (repulsion). The acceleration is then zero in these positions viz. $r = \alpha$, $r = \frac{2A\alpha}{(A-1)}$, $r = \infty$. In the first interval there will be a repulsion, in the second attraction; within either interval there is an extreme. If $A > 1$ then, according to (18), r cannot be greater than $\frac{A\alpha}{(A-1)}$. Then the motion is that of a particle first going away from the centre and then returning when $r = \frac{A\alpha}{(A-1)}$. The value $\frac{2A\alpha}{(2A-1)}$, of r, for which the acceleration becomes zero, is smaller than $\frac{A\alpha}{(A-1)}$. The particle ascends (during which there is first repulsion); at a given moment the acceleration becomes zero for $r = \frac{2A\alpha}{(2A-1)}$; then we get attraction, which for $r = \frac{A\alpha}{(A-1)}$ has exhausted the motion and makes it return; the acceleration of the reversed motion is first positive, then becomes negative for $r = \frac{2A\alpha}{(2A-1)}$ and the motion stops (infinitely slowly) for $r = \alpha$. In case that A lies between 0 and $\frac{1}{2}$ so that r can have all values, there is no point where the acceleration becomes zero. According to (20) there is then always repulsion; the velocity is maximum at an infinite distance viz., according to (18), $\sqrt{1-A}$ which lies between $\frac{1}{2}\sqrt{2}$ and 1.

8. We now return to the general case, where neither $\dot{r}$ nor $\dot{\varphi}$ are continually zero. We must then take equations (13) and (14) as a starting point; by eliminating dt we find

$$\frac{1}{1-\frac{\alpha}{r}} - \frac{B^2}{r^4} \cdot \frac{1}{1-\frac{\alpha}{r}} \left(\frac{dr}{d\varphi}\right)^2 - \frac{B^2}{r^2} = A. \qquad (21)$$

Expressing $d\varphi$ in r and dr we obtain

$$d\varphi = \frac{B dr}{r^2 \sqrt{1 - \left(A + \frac{B^2}{r^2}\right)\left(1 - \frac{\alpha}{r}\right)}}.$$

Putting now $\frac{\alpha}{r} = x$, we get

$$d\varphi = \frac{-dx}{\sqrt{x^3 - x^2 + \frac{A\alpha^2}{B^2} x + \frac{(1-A)\alpha^2}{B^2}}}.$$

So φ becomes an elliptic integral in the variable r, and r therefore an elliptic function of φ. Of

$$x^3 - x^2 + \frac{A\alpha^2}{B^2} x + \frac{(1-A)\,\alpha^2}{B^2} = 0$$

let x_1, x_2, x_3 be the roots, so that

$$x_1 + x_2 + x_3 = 1,\ x_1x_2 + x_2x_3 + x_3x_1 = \frac{A\alpha^2}{B^2},\ x_1x_2x_3 = \frac{(A-1)\,\alpha^2}{B^2}, \qquad (22)$$

then we can introduce as constants of integration the quantities x_1, x_2, x_3 (connected by the relation $x_1 + x_2 + x_3 = 1$) instead of A and B.

If we now introduce a new variable

$$z = x - \frac{1}{3}$$

putting

$$e_1 = x_1 - \frac{1}{3},$$
$$e_2 = x_2 - \frac{1}{3},$$
$$e_3 = x_1 - \frac{1}{3},$$

we obtain

$$d\varphi = \frac{-\alpha z}{\sqrt{(z-e_1)\,(z-e_2)\,(z-e_3)}}, \tag{23}$$

and we have

$$e_1 + e_2 + e_3 = 0, \tag{24}$$

Now, introducing the ρ-function with the roots e_1, e_2, e_3, we get

$$z = \rho\big(\tfrac{1}{2}\varphi + C\big),$$

where C is a constant of integration, which may be complex; the real part is without signification as it only determines the direction in which φ will be zero. We take

$$z = \rho\big(\tfrac{1}{2}\varphi + is\big), \tag{25}$$

and then find

$$\tfrac{\alpha}{r} = \tfrac{1}{3} + \rho\big(\tfrac{1}{2}\varphi + is\big). \tag{26}$$

From (14) now follows

$$Bdt = \frac{r^2}{1-\frac{\alpha}{r}} = \frac{\alpha^2 d\varphi}{x^2\,(1-x)} = -\alpha^2 \frac{dx}{x^2\,(1-x)\,\sqrt{(x-x_1)\,(x-x_2)\,(x-x_3)}}$$

or

$$\frac{B}{\alpha^2}dt = \frac{-dz}{\big(z+\frac{1}{3}\big)^2\,\big(\frac{2}{3}-z\big)\,\sqrt{(z-e_1)\,(z-e_2)\,(z-e_3)}}. \tag{27}$$

The problem under consideration gives rise to four constants of integration; two of which are e_1 and e_2, the two others s (which can have only particular values) and a constant which arises after integration of (27) and is of no consequence as it only determines the moment at which $t = 0$.

From (27) it now follows immediately that the particle can never reach sphere $r = \alpha$. For, if r became α, then z became $\frac{2}{3}$; (27) shows that this would require an infinitely long time. Sphere $r = \alpha$, therefore, is never reached.

It also follows from (27) that an infinitely long time is required for z to reach $-\frac{1}{3}$. This is not at all strange, $z = -\frac{1}{3}$ corresponding to $r = \infty$. It may occur (if two e's coincide) that there is still another value of r which cannot be attained, but is gradually approached; we will treat this case where it occurs.

9. Let us now first consider *the case* $e_1 = e_2 = e_3 = 0$.

Equation (23) becomes

$$d\varphi = \frac{\alpha z}{z^{\frac{3}{2}}}, \tag{28}$$

so

$$\varphi = \frac{2}{\sqrt{z}} = \frac{2}{\sqrt{\frac{\alpha}{r} - \frac{1}{3}}}. \tag{29}$$

The value 3α of r, corresponding to $z = 0$, is, as is seen from (27), a value which is not attained. (29) shows that the motion takes place in a spiral which, extending to circle $r = \alpha$, making there with the radius a finite angle, and, turning an infinite number of times, approaches to circle $r = 3\alpha$ on the inside. The particle can never get out of sphere $r = 3\alpha$ and a motion such that the particle were from the beginning outside sphere $r = \alpha$ (and such that $e_1 = e_2 = e_3 = 0$), is impossible according to (28), as $\left(\frac{dz}{d\varphi}\right)^2$ should be negative.

When r approaches to 3α then φ approaches to $\frac{1}{3\alpha\sqrt{6}}$ and consequently the velocity to $\frac{1}{\sqrt{6}}$.

10. We now come to the case of two e's being equal and different from the third. Calling (the three e's being real) the greatest e_1 the smallest e_3, we have two cases, viz.

$$e_2 = e_3 = -\frac{1}{2}e_1, \;\; e_1 = e_2 = -\frac{1}{2}e_3.$$

We first turn to *the case* $e_2 = e_3 = -\frac{1}{2}e_1$.

Excluding as before the interior of sphere $r = \alpha$, r must be $> \alpha$, so $z < \frac{2}{3}$. We put $e_2 = e_3 = -\alpha^2$, $e_1 = 2\alpha^2$; α be positive. Then (23) passes into

$$d\varphi = \frac{-dz}{(z + \alpha^2)\sqrt{z - 2\alpha^2}}.$$

It is seen that z must be greater than $2\alpha^2$, and, as z must be smaller that $\frac{2}{3}$, we must have

$$2\alpha^2 < \frac{2}{3}. \tag{30}$$

If $2\alpha^2 = \frac{2}{3}$, the particle is at rest on sphere $r = \alpha$.

Now putting $z = 2\alpha^2 + y^2$ we get

$$\frac{1}{2}d\varphi = \frac{-dy}{y^2 + 3\alpha^2},$$

and so

$$y = -\alpha\sqrt{3}\; tg\left(\frac{1}{2}\alpha\varphi\sqrt{3}\right).$$

This gives us

$$r = \frac{\alpha}{\frac{1}{3} + 2\alpha^2 + 3\alpha^2\; tg^2\left(\frac{1}{2}\alpha\varphi\sqrt{3}\right)}. \tag{31}$$

The case $\alpha = 0$ has been discussed in 9: we therefore put $\alpha \neq 0$. When $\varphi = 0$, $r = \alpha : \left(\frac{1}{3} + 2\alpha^2\right)$, i.e. a value between $r = \alpha$ and $r = 3\alpha$. When φ approaches to $\pi : \alpha\sqrt{3}$ (a value which, from (30), exceeds π) r should approach to zero, according to (31). But first r must become equal to α, viz. when φ becomes

$$\varphi = \varphi_o = \frac{2}{\alpha\sqrt{3}} \cdot arc\ tg \frac{\sqrt{2-6\alpha^2}}{3\alpha}$$

and for this, according to (27), an infinite time is required as then $z = \frac{2}{3}$. So the motion is as follows: φ changes from $-\varphi_o$ to φ_o, corresponding to $r = \alpha$. The greatest value of r is reached at the moment when $\varphi = 0$, viz.

$$r = \frac{\alpha}{\frac{1}{3} + 2\alpha^2} < 3\alpha;$$

when $\varphi = -\varphi_o$ (as well as when $\varphi = \varphi_o$) r becomes α. If r approaches to zero, φ_o increases indefinitely and the motion approaches more and more to that which has been discussed in 9.

11. *The case* $e_1 = e_2 = -\frac{1}{2}e_3$.
Put $e_1 = e_2 = \alpha^2$, $e_3 = -2\alpha^2$, then (23) passes into

$$d\varphi = \frac{-dz}{(z-\alpha^2)\sqrt{z+2\alpha^2}}. \tag{32}$$

As $z > -2\alpha^2$, we may put $z = -2\alpha^2 + y^2$. Then we get

$$d\varphi = -\frac{2\ dy}{y^2 - 3\alpha^2}.$$

Now, if $z > \alpha^2$, and therefore $y^2 > 3\alpha^2$, we get

$$y = \alpha\sqrt{3}\ cotgh\left(\frac{1}{2}\alpha\varphi\sqrt{3}\right),$$

and

$$r = \frac{\alpha}{\frac{1}{3} - 2\alpha^2 + 3\alpha^2\ cotgh^2\left(\frac{1}{2}\alpha\varphi\sqrt{3}\right)}. \tag{33}$$

If, on the contrary, $z < \alpha^2$ and consequently $y^2 < 3\alpha^2$,

$$y = \alpha\sqrt{3}\ tgh\left(\frac{1}{2}\alpha\varphi\sqrt{3}\right),$$

and so

$$r = \frac{\alpha}{\frac{1}{3} - 2\alpha^2 + 3\alpha^2\ tgh^2\left(\frac{1}{2}\alpha\varphi\sqrt{3}\right)}. \tag{34}$$

z cannot pass α^2 and must moreover lie between $-\frac{1}{3}$ and $\frac{2}{3}$.

So we have the following cases:

A. $\alpha^2 \geq \frac{2}{3}$. z lies between $-\frac{1}{3}$ and $\frac{2}{3}$; formula (34) holds: r varies between ∞ and α; the first value is attained for

$$\varphi = \varphi_1 = \frac{1}{\alpha\sqrt{3}}\ \log\frac{\alpha\sqrt{3} + \sqrt{2\alpha^2 - \frac{2}{3}}}{\alpha\sqrt{3} - \sqrt{2\alpha^2 - \frac{1}{3}}},$$

and the second for

$$\varphi = \varphi_2 = \frac{1}{\alpha\sqrt{3}}\ \log \frac{\alpha\sqrt{3} + \sqrt{2\alpha^2 + \frac{2}{3}}}{\alpha\sqrt{3} + \sqrt{2\alpha^2 + \frac{2}{3}}}.$$

An infinitely long time is required to reach either position.

B. $\alpha^2 < \frac{2}{3}; z$ between α and $\frac{2}{3}$. formula (33) must be applied; r varies between $\alpha : \left(\frac{1}{3} + \alpha^2\right)$ and α; φ then changes from ∞ to

$$\varphi = \varphi_3 = \frac{1}{\alpha\sqrt{3}}\ \log \frac{\sqrt{2\alpha^2 + \frac{2}{3}} + \alpha\sqrt{3}}{\sqrt{2\alpha^2 + \frac{2}{3}} - \alpha\sqrt{3}}.$$

The orbit comes from $r = \alpha$ and approaches in a spiral to circle $r = \alpha : \left(\frac{1}{3} + \alpha^2\right)$.

C. $\frac{1}{6} \leq \alpha^2 < \frac{2}{3}; z$ between $-\frac{1}{3}$ and α^2. Formula (34) now holds; r varies between ∞ and $\alpha : \left(\frac{1}{3} + \alpha^2\right)$; φ changes from φ_1 to ∞. The orbit comes from infinity and turns in a spiral round the circle $r = \alpha : \left(\frac{1}{3} + \alpha^2\right)$ which lies between circle 2α and circle α.

D. $\alpha^2 < \frac{1}{6}; z$ between $-2\alpha^2$ and α^2. Formula (34) must be applied; r varies between $\alpha : \left(\frac{1}{3} - 2\alpha^2\right)$ and $\alpha : \left(\frac{1}{3} + \alpha^2\right)$; φ changes from 0 to ∞. The orbit is a spiral, coming from circle $\alpha : \left(\frac{1}{3} - 2\alpha^2\right)$, which may have any radius $> 3\alpha$, and approaching in a infinite number of turnings to circle $\alpha : \left(\frac{1}{3} + \alpha^2\right)$, which lies between circle 2α and circle 3α.

12. Now we will suppose the roots e_1, e_2, e_3 to be all different. As regards these roots, we may then distinguish two main cases, viz. the case of three real roots and the case of one real and two conjugate complex roots. In the first case we put $e_1 > e_2 > e_3$, in the second e_2 be the real root and the imaginary part of e_1 be positive. In either case we put, as usual, $e_1 = \rho\omega_1, e_2 = \rho\omega_2, e_3 = \rho\omega_3$, with $\omega_2 = \omega_1 + \omega_3$ (not $-\omega_1 - \omega_3$).

The three roots are real. the only values possible for $i\ s$ in equation (25) now are 0 and ω_3 (or congruent values). In the first case z varies from ∞ to e_1 and from e_1 to ∞, while φ changes from 0 to $2\omega_1$ and from $2\omega_1$ to $4\omega_1$. One must, however, remember that, according to (27), z may not exceed the values $-\frac{1}{3}$ and $\frac{2}{3}$ (i.e. $r = \infty$ and $r = \alpha$), but must remain between them. So if $e_1 > \frac{2}{3}$, it is impossible for $i\ s$ to be zero. If $e_1 < \frac{2}{3}$, z varies between e_1 and $\frac{2}{3}$ and so r between $\frac{\alpha}{\left(\frac{1}{3} + e_1\right)}$ and α. This case corresponds to 10 and 11 B into which it passes when $e_2 = e_3 = -\frac{1}{2}e_1$ and when $e_2 = -2e_1$.

In the other case ($i\ s \equiv \omega_3$) z varies from e_3 to e_2 and from e_2 to e_3, while φ changes from 0 to $2\omega_1$ and from $2\omega_1$ to $4\omega_1$. There are various cases:

A. $e_2 \geq \frac{2}{3}$. z varies between $-\frac{1}{3}$ and $\frac{2}{3}$, φ between φ_1 and φ_2 for which

$$-\tfrac{1}{3} = \rho\left(\tfrac{1}{2}\varphi_1 + \omega_3\right) \text{ and } \tfrac{2}{3} = \rho\left(\tfrac{1}{2}\varphi_2 + \omega_3\right);$$

φ_1 lies between 0 and ω_1, φ_2 between 0 and $2\omega_1$ ($\varphi_2 > \varphi_1$). r changes between ∞ and α. This case corresponds to 11A and passes into it for $e_2 = e_1 = -\frac{1}{3}$.

B. $e_3 \leq -\frac{1}{3}, e_2 < \frac{2}{3}$. z varies between $-\frac{1}{3}$ and e_2, φ between φ_1 and $2\omega_1$; r changes from ∞ to $\frac{\alpha}{\left(\frac{1}{3}+e_2\right)}$, a value between 2α and α. This corresponds to **11C**, in which it passes for $e_1 = e_2, \omega_1$ then becoming infinite.

C. $e_3 > -\frac{1}{3}, e_2 < \frac{1}{6}$. z varies between e_3 and e_2, φ between $-\infty$ and $+\infty$; r changes from $\frac{\alpha}{\left(\frac{1}{3}+e_3\right)}$, which may have all values $> 3\alpha$, to $\frac{\alpha}{\left(\frac{1}{3}+e_2\right)}$, which may have all values between 2α and α. The case corresponds to **11D**, in which it passes for $e_1 = e_2 > 0$; if $e_2 < 0$ there is no corresponding degenerated case.

Two roots are conjugate complex. The value which in (25) is possible for *is* is 0. then z varies from ∞ to e_2 and back. So if $e_2 \geq \frac{2}{3}$ this case is impossible. If $-\frac{1}{3} < e_2 < \frac{2}{3}, z$ varies between $\frac{2}{3}$ and e_2, φ between a value φ_3 for which

$$\rho\left(\tfrac{1}{2}\varphi_3\right) = \tfrac{2}{3}$$

(situated between 0 and $2\omega_3$) and $4\omega_2 - \varphi_3$. r changes from α to $\frac{\alpha}{\left(\frac{1}{3}+e_2\right)}$, which may have any value $> \alpha$, and then returns to α. This case can pass into **10**, if e_1 and e_3 approach to the same negative value; and, if e_2 becomes negative, it may divide itself into **11B** on the one hand and **11C** or **11D** on the other (**11C** if $e_2 < -\frac{1}{3}$, **11D** if $e_2 > -\frac{1}{3}$,).

We now have a survey of all possible motions. We must, however, remark that not all the motions take place with a velocity smaller than that of light, as in case of some of them (e.g. **11A** and **12A**) **A** and so also **L** is negative. We have not separately mentioned all those cases. In **11** e.g. $\alpha^2 < \frac{1}{3}$, means that the velocities are smaller than that of light. In **12** for that purpose $e_1e_2 + e_2e_3 + e_3e_1$ has to be $> -\frac{1}{3}$.

13. It is now necessary to consider the place taken up in this survey by the well-known motions of the planets and comets. These motions all take place with small velocities; we will call a quantity such as the square of the velocity of a planet, a quantity of the first order. In NEWTON'S theory, which accounts very exactly for the motions, $\alpha : r$ is found to be of the same order as the square of a velocity; this we take from NEWTON'S theory. In (13) A must then be a quantity, differing little from 1; we represent it by

$$A = 1 + \frac{\mu\alpha}{\lambda^2}.$$

In (14) B is a quantity of order $\frac{1}{2}$. We represent is by

$$B = \sqrt{\alpha} : \lambda$$

and take λ positive. The constants λ and μ then take the places of A and B. If we substitute these constants in (21), this equation becomes

$$\frac{\lambda^2}{r} \cdot \frac{1}{1 - \frac{\alpha}{r}} \frac{1}{r^4} \left(\frac{dr}{d\varphi}\right)^2 - \frac{1}{r^2} = \mu. \tag{35}$$

The constants λ and μ are moderately great. The formula passes into the corresponding one of NEWTON'S theory, if we put $\alpha = 0$. We then obtain

$$\frac{\lambda^2}{r} - \frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 - \frac{1}{r^2} = \mu. \tag{36}$$

The equation gives rise to an ellipse, if μ is positive, to a parabola if $\mu = 0$, to a hyperbola if μ is negative. In NEWTON'S theory $4\mu < \lambda^4$. In consequence of the introduction of the constants λ and μ the equations pass into

$$x_1 + x_2 + x_3 = 1,\ x_1x_2 + x_2x_3 + x_3x_1 = \alpha\left(\lambda^2 + \mu\alpha\right),\ x_1x_2x_3 = \mu\alpha^2. \tag{37}$$

We see from these that the roots x_1, x_2, x_3 approach very nearly to $1, 0, 0$. The quantity $\alpha\left(\lambda^2 + \mu\alpha\right)$ is positive. Because $\mu < \frac{1}{4}\lambda^4$ the roots prove to be all real. x_1 is somewhat smaller than 1, about $\alpha\lambda^2; x_2$ and x_3 are of the order of α; they are both positive if μ is positive, else they have opposite signs; x_3 becomes zero if $\mu = 0$. We will therefore put

$$x_1 = 1 - 2\alpha m,$$

$$x_2 = \alpha\left(m + n\right),$$

$$x_3 = \alpha\left(m - n\right).$$

Now $x_1 + x_2 + x_3 = 0$ as it ought to be; if $n < m$ we have to deal with the quasi-elliptic motion, if $n > m$ with the quasi-hyperbolic, if $n = m$ with the quasi-parabolic. The constants m and n take the places of λ and μ. We obtain

$$e_1 = \frac{2}{3} - 2\alpha m,$$

$$e_2 = -\frac{1}{3} + \alpha\left(m + n\right),$$

$$e_3 = -\frac{1}{3} + \alpha\left(m - n\right). \tag{38}$$

In (22) and (26) we now take, in the case of elliptic motion, $is = \omega_3$, as φ increases indefinitely, z remaining finite. In the case of the parabolic and hyperbolic motion r becomes infinite and so $z = -\frac{1}{3}; z$ moves bewteen e_3 and e_2 and again $is = \omega_3$. So (26) becomes

$$\tfrac{\alpha}{r} = \tfrac{1}{3} + \wp\left(\tfrac{1}{2}\varphi + \omega_3\right).$$

Now we have the formula

$$\wp\left(\tfrac{1}{2}\varphi + \omega_3\right) = e_3 + \left(e_1 - e_3\right)\left(e_2 - e_3\right) \div \left(\wp\tfrac{1}{2}\varphi - e_3\right),$$

and so

$$\tfrac{\alpha}{r} = \tfrac{1}{3} + e_3 + \left(e_1 - e_3\right)\left(e_2 - e_3\right) \div \left(\wp\tfrac{1}{2}\varphi - e_3\right),$$

or from (38)

$$\tfrac{1}{r} = m - n + 2n\,(e_1 - e_3) \div (\wp \tfrac{1}{2}\varphi - e_3). \tag{39}$$

This is the equation of the orbit required. If we now let α become zero, e_3 and e_2 coincide, $e_1 - e_3$ becomes 1, and the $\wp$-function degenerates. We then obtain

$$\frac{1}{r} = m - n + 2n\sin^2\frac{1}{2}\varphi = m - n\cos\varphi \tag{40}$$

and this equation shows once more that, if $\alpha \neq 0$, for $n < m$ the motion is (quasi-)elliptic, for $m > n$ (quasi)-hyperbolic, for $n = m$ (quasi)-parabolic. For $n = 0$ it is circular, also if α is not supposed to be zero. The elliptic case is case 12 C, the hyperbolic is 12 B, the parabolic is 12 B, e_3 being supposed to be $-\frac{1}{3}$ there.

14. Let us now examine the motion of the planets a little more in detail. Equation (39) shows that $4\omega_1$ is the period; as the $\wp$-function is almost degenerated we may take

$$4\omega_1 = \frac{4\pi}{\sqrt{e_1 - e_3} + \sqrt{e_1 - e_2}}. \tag{41}$$

A further approximation is not necessary as, after expanding the roots in a series of ascending powers of α, the terms of degree 0 and 1 do not change any longer. From (41) it follows in this way

$$4\omega_1 = 2\pi\left(1 + \frac{3}{2}\alpha m\right) = 2\pi + 3\alpha m\pi.$$

Now (39) shows that $m - n$ is the smallest, $m + n$ the greatest value of $\frac{1}{r}$. From this or from (40) it follows that m is the reciprocal value of the parameter p of the orbit and $\frac{n}{m}$ represents the eccentricity; so

$$e = \frac{n}{m},\ p = \frac{1}{m}. \tag{42}$$

This gives for the motion of the perihelion per period $\frac{3\alpha\pi}{p}$ corresponding to the value calculated by EINSTEIN.

To conclude we will calculate the periodic time. From (14) follows

$$Bdt = \frac{r^2 d\varphi}{1 - \frac{\alpha}{r}}.$$

If we put in this $\alpha = 0$ we obtain the corresponding equation of NEWTON'S theory; we may therefore expand the denominator and obtain as a first approximation

$$Bdt = r^2\left(1 + \frac{\alpha}{r}\right) d\varphi = r^2 d\varphi + \alpha r d\varphi. \tag{43}$$

We must now substitute for r the value taken from (39). Let us for a moment introduce the elliptic function sn with the modulus k, defined by

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{2\alpha n}{1 - 3\alpha m + \alpha n}, \tag{44}$$

(39) passes into

$$\frac{1}{r} = m - n + 2n\ sn^2\ \frac{1}{2}\varphi\sqrt{e_1 - e_3}; \tag{45}$$

k^2 is of the first order, and consequently very small. If we put

$$\sin\psi = sn\ \frac{1}{2}\varphi\sqrt{e_1 - e_2}, \tag{46}$$

we find by differentiation

$$\cos\psi\ d\psi = \frac{1}{2}\sqrt{e_1 - e_3}\sqrt{\left(1 - \sin^2\psi\right)\left(1 - k^2\sin^2\psi\right)}d\varphi$$

or

$$\frac{1}{2}\sqrt{e_1 - e_3}\ d\varphi = \frac{d\varphi}{\sqrt{1 - k^2\sin^2\psi}}.$$

Now as (45) passes into

$$\frac{1}{r} = m - n + 2n\sin^2\psi,$$

(43) becomes

$$\frac{1}{2}B\sqrt{e_1 - e_3}\ dt = \frac{d\psi}{\left(m - n + 2n\sin^2\psi\right)^2\sqrt{1 - k^2\sin^2\psi}} +$$

$$+ \frac{\alpha d\psi}{\left(m - n + 2n\sin^2\psi\right)\sqrt{1 - k^2\sin^2\psi}}.$$

If $\alpha = k = 0$ we pass into NEWTON'S theory. So in the first fraction we may expand the denominator and neglect k^4, etc., and in the second fraction we may put $k = 0$. Putting $k^2 = 2\alpha n$ in the first fraction we obtain

$$\frac{1}{2}B\sqrt{e_1 - e_3}\ dt = \frac{1 + \alpha n\sin^2\psi}{\left(m - n + 2n\sin^2\psi\right)^2}d\psi + \frac{\alpha d\psi}{\left(m - n + 2n\sin^2\psi\right)}$$

$$= \frac{1 - \frac{1}{2}\alpha\left(m - n\right)}{\left(m - n + 2n\sin^2\psi\right)^2}d\psi + \frac{\frac{3}{2}\alpha d\psi}{\left(m - n + 2n\sin^2\psi\right)}. \tag{47}$$

From the values of x_1, x_2, x_3 we get, considering (22),

$$B\sqrt{e_1 - e_3} = \sqrt{\frac{\alpha}{2m}}\left(\frac{1 - 3\alpha m + \alpha n}{1 - 2\alpha m + \alpha^2\left(m^2 - n^2\right)}\right)^{\frac{1}{2}}.$$

We may write

$$B^{-1}\left(e_1-e_3\right)^{-\frac{1}{2}}=\sqrt{\frac{2m}{\alpha}}\left[1+\frac{1}{2}\alpha\left(m-n\right)\right]$$

and so (47) passes into

$$\frac{1}{2}\sqrt{\frac{\alpha}{2m}}dt=\frac{d\psi}{\left(m-n+2n\sin^2\psi\right)^2}+\frac{\frac{3}{2}\alpha d\psi}{\left(m-n+2n\sin^2\psi\right)}.$$

We will call the time in which r is periodic the periodic time; it is the time in which φ increases by $4\omega_1$ and ψ by π. So

$$\frac{1}{2}\sqrt{\frac{\alpha}{2m}}T=\int_0^\pi\frac{d\psi}{\left(m-n+2n\sin^2\psi\right)^2}+\frac{3}{2}\alpha\int_0^\pi\frac{d\psi}{\left(m-n+2n\sin^2\psi\right)}=$$

$$=\frac{\pi m}{\left(m^2-n^2\right)^{\frac{3}{2}}}+\frac{\frac{3}{2}\alpha\pi}{\left(m^2-n^2\right)^{\frac{1}{2}}}.$$

In connection with (42) we get from this, a representing half the major axis:

$$\frac{\sqrt{\alpha}}{2\pi\sqrt{2}}T=a^{\frac{3}{2}}+\frac{3}{2}\alpha a^{\frac{1}{2}},$$

or with the same degree of approximation

$$\frac{\sqrt{\alpha}}{2\pi\sqrt{2}}T=\left(a+\alpha\right)^{\frac{3}{2}}.$$

We so obtain KEPLER'S third law

$$\frac{\left(a+\alpha\right)^3}{T^2}=\frac{\alpha}{8\pi^2}. \tag{48}$$

We can also ask after the time required by φ to increase by 2π. This time depends on the place from which the planet starts; it is greatest for the perihelion, smallest for the aphelion. As a mean value of all these times we may consider

$$T_1=T\left(1-\frac{3\alpha}{2p}\right).$$

For this time KEPLER'S third law becomes

$$\left(a-\frac{\alpha e^2}{1-e^2}\right)^3:T_1^2=\frac{\alpha}{8\pi^2}.$$

This deviates from KEPLER'S law less than (48).