THE TOTAL SPACE-TIME OF A POINT-MASS WHEN $\Lambda \neq 0$, AND ITS CONSEQUENCES FOR THE LAKE-ROEDER BLACK HOLE

LEONARD S. ABRAMS

Abstract. Singularities associated with an incomplete space-time (S) are not well-defined until a boundary is attached to it. Moreover, each boundary (B) gives rise to a different singularity structure for $S \cup B$, the resulting "total" space-time (TST). Since S is compatible with a variety of boundaries, it therefore does not represent a unique universe, but instead corresponds to a family of universes, one for each possible boundary.

It is shown that in the case of Weyl's space-time for a point-mass with nonzero Λ, the boundary which he attached to it is invalid, and when the correct one is attached, the resulting TST is inextendible. This implies that the Lake-Roeder black hole cannot be produced by gravitational collapse.

1. Introduction

Let $S = (M, g)$ denote an incomplete, not necessarily inextendible space-time. It is well known [1] that the singularities of S are not completely specified by (M, g) alone, and that in order to achieve such a specification a boundary (B, say) must be attached to it. [The resulting object, $T = S \cup B$, will be termed a "total" space-time (TST).]

Moreover, thanks to the work of a dozen or so investigators in the sixties and seventies (e.g. Refs. [2], [3], [4]), it is now also well-known that an incomplete space-time is compatible with a variety of boundaries, each of which gives rise to a different singularity structure for the resulting TST.

Taken together, these two facts show that an incomplete space-time by itself does not represent a unique universe - rather, it corresponds to a family of universes, one for each of its possible boundaries. Changing the boundary attached to S changes the universe represented by the resulting TST [5].

Let us now apply the foregoing considerations to a universe (U_0) consisting of a single point-mass and a nonzero cosmological constant (Λ). As is well-known, the space-time (S_0) for U_0 was first obtained by Weyl [6]. Unfortunately, Weyl's derivation (Section 2) involved a tacit assumption which

Key words and phrases. Incomplete space-times; Boundaries; Black holes.
PACS: 97.60 Lf; 04.20 Jb.
Published in Physica A **227** (1996) 131-140.
Phone:(818)883-6298.

is shown here (Section 3) to be invalid. When this assumption is eliminated (Section 4), the resulting space-time (termed "Stavroulakis", since its metric is a special case of one found by him [7]) is isometric, and hence equivalent [8], to S_0. Thus, in this respect the invalidity of Weyl's assumption is harmless. However, Stavroulakis' (and Weyl's) space-time is timelike incomplete, so that it cannot represent any universe, and thus in particular does not represent U_0. The question therefore arises: What boundary must be attached to S_0 in order that the resulting TST represent U_0? Since in Einstein's relativity point-masses are necessarily singularities of the field, and since U_0 has no sources other than the single point mass, the answer is immediate: the boundary must be a line through the point-mass. [Equivalently, in each spatial section the boundary must consist of a point at the location of the point-mass.] It is here that Weyl's assumption gave rise to a fatal flaw, since it automatically attached a boundary consisting of a two-sphere at the location of the point-mass. (See Section 7.2.)

It is then shown (Section 5) that with the correct boundary, Stavroulakis' TST is inextendible and contains no black hole. As a result (Section 6), for $\Lambda \neq 0$ no black hole can be formed when a spherically symmetric, uncharged, nonrotating star undergoes gravitational collapse. Lacking both a valid derivation for a specific universe and a plausible model for its production, it follows that the black hole found by Lake and Roeder [9] and by Laue and Weiss [10] is nothing more than an artifact of a historical error.

It should be emphasized that since a TST is a space-time-with-boundary, the criteria for equivalence and extendibility [11] are necessarily different from those applicable to space-times. Specifically, it follows from their definition that equivalence of TST requires not only that their interiors be isometric, but also that their boundaries be homeomorphic. Likewise, extendibility of a TST requires not only that its interior be isometric to the proper open subset of another space-time, but also that its boundary be preserved under the mapping, i.e., that the image of B be homeomorphic to B itself.

2. Weyl's derivation

Weyl's derivation of the metric of an uncharged point-mass when $\Lambda \neq 0$ was as follows: Starting from the most general expression [12] for the static metric which is spherically-symmetric about the location of the point-mass (taken, without loss of generality, to be $x = y = z = 0$), viz.:

$$g(r) = A(r)dt^2 - B(r)dr^2 - C(r)d\Omega^2, \; A, B, C > 0, \tag{2.1}$$

where r, θ, ϕ are quasi-spherical polar coordinates [i.e., $r = (x^2 + y^2 + z^2)^{1/2}$, etc.], he introduced a new radial coordinate via:

$$r^* = [C(r)]^{1/2}, \tag{2.2}$$

which transforms (2.1) into

$$g^*(r^*) = A^*(r^*)dt^2 - B^*(r^*)dr^{*2} - r^{*2}d\Omega^2, \tag{2.3}$$

and, of course, assigns to the location of the point-mass the value

$$r_0^* = [C(0+)]^{1/2}. \tag{2.4}$$

He then solved for A^*, B^* using a variational principle equivalent to the vacuum field equations, obtaining [13]

$$A^* = A^*_{\mathrm{w}} \equiv 1 - \alpha/r^* - \Lambda r^{*2}/3, \tag{2.5}$$

$$B^* = B^*_{\mathrm{w}} \equiv 1/A^*_{\mathrm{w}}, \tag{2.6}$$

where α is a constant.

Unfortunately, the introduction of r^* as a coordinate creates something of a problem - since $C(0+)$ is unknown, there is no way to determine the value of r_0^*, the location of the point-mass (cf. Ref. [14]). This was overlooked by Weyl, who had tacitly assumed [15] that the point-mass' location was given by $r^* = 0$. As follows from (2.4), this can only be true if $b^2 \equiv C(0+) = 0$.

3. The invalidity of Weyl's assumption

In order to determine whether this assumption is valid, one could substitute (2.1) into the vacuum field equations, solve for A, B, C, and then see whether $C(0+) = 0$ ia admissible (cf. [14]). However, there is a simpler way - all one need do is return to the r-coordinate system, in which the location of the point-mass is known, by using (2.2). So doing changes (2.3) into (2.1), and (2.5) and (2.6) into:

$$A = 1 - \alpha/\sqrt{C} - \Lambda C/3, \tag{3.1}$$

$$B = C'^2/(4AC), \tag{3.2}$$

respectively, where at this stage C is any positive analytic function of r for $r > 0$.

It follows by inspection of (3.2) that in order to insure the positivity of B, it is necessary and sufficient that C'^2 be nonvanishing for $r > 0$. This in turn requires that either $C' > 0$ for $r > 0$ or $C' < 0$ for $r > 0$. Since the metric must tend to that of Schwarzschild as $\Lambda \to 0$, the only possible choice is:

$$C' > 0 \text{ for } r > 0. \tag{3.3}$$

The constraints on C required to assure the positivity of A cannot be determined by mere inspection of (3.1), since the behavior of a cubic is involved. The analysis is relegated to Appendix A, and only the results are given here:

$$\text{For } \Lambda < 0 \quad C(0+) \equiv b^2 \geq C_0 > 0, \tag{3.4}$$

$$\text{For } 0 < \Lambda < \Lambda_0 \quad C_3 \geq b^2 \geq C_2 > 0 \; [\Lambda_0 \equiv 4/(9\alpha^2)], \tag{3.5}$$

$$\text{For } \Lambda_0 \leq \Lambda \quad \text{No metrics with } A \text{ satisfying (3.1) exist.}$$

(The values of C_0, C_2 and C_3 are given in Appendix A.) In view of the foregoing, it follows that the necessity that A be positive for $r > 0$ requires that $b^2 > 0$, which renders Weyl's assumption invalid for all admissible values of Λ.

4. Additional restrictions on C

As shown by Doughty [16], the locally measured acceleration of an uncharged test particle in a gravitational field with metric given by (2.1) is

$$a = |A'|/(2A\sqrt{B}). \tag{4.1}$$

Substituting from (3.1) and (3.2), this becomes

$$a = |\alpha/(2C^{3/2}) - \Lambda/3|(C/A)^{1/2}, \tag{4.2}$$

which in turn tends to

$$a_0 = |\alpha/(2b^3) - \Lambda/3|b/\sqrt{A(0+)} \quad \text{as} \quad r \to 0. \tag{4.3}$$

As pointed out in Ref. [14], the value of a_0 is a scalar differential invariant of the space-time. Thus, different values of a_0 give rise to inequivalent space-times. Here, as in Ref. [14], we shall take a_0 to have its Newtonian value, infinity. In view of the nonzero values of b found to be required in Section 3, it follows from (4.3) that the only way to make a_0 infinite is to choose $A(0+) = 0$. As indicated in Appendix A, this in turn requires that:

$$b = \sqrt{C_0} \quad \text{for } \Lambda < 0, \tag{4.4}$$

$$= \sqrt{C_2} \quad \text{for } 0 < \Lambda < \Lambda_0. \tag{4.5}$$

The final restriction on C is obtained by considering the limiting behavior of the point-mass space-time as $r \to \infty$. In the $\Lambda = 0$ case, this behavior was determined by requiring that as the proper distance from the point-mass became unboundedly large, the metric must approach that in which no point-mass is present - i.e., Minkowski's. Here, however, in each spatial section ($t =$ constant), the integral

$$R_{\max} \equiv \int_0^\alpha \sqrt{B} dr = 1/2 \int_{b^2}^{k^2} (1/\sqrt{AC}) dC \tag{4.6}$$

may be either finite or infinite, depending on whether k [$\equiv \sqrt{C(\infty)}$] is finite or infinite. In the former case, which is always true when $0 < \Lambda < \Lambda_0$ (see Appendix A), there are no events which are at infinite proper distances from the point-mass, so that no asymptotic condition corresponding to that for $\Lambda = 0$ exists. However, it is easy to show (Appendix B) that if $k < k_{\max} \equiv \sqrt{C_3}$, then the associated TST is extendible to one for which $k = k_{\max}$, so that for $0 < \Lambda < \Lambda_0$ the only viable choice of k is

$$\sqrt{C(\infty)} = k_{\max}. \tag{4.7}$$

Similarly, if for $\Lambda < 0$ the limiting value of C as $r \to \infty$ were taken to be finite, then the associated TST would be extendible to one for which $C(\infty)$

is infinite (see Appendix B). Thus, for $\Lambda < 0$ the only viable choice of k is given by

$$\sqrt{C(\infty)} = \infty. \tag{4.8}$$

In this case, since $R_{\max}$ is infinite, then for large r the influence of the point-pass on the space-time geometry must necessarily vanish, just as in the $\Lambda = 0$ case. Thus, the metric must approach that for which no point-mass is present. Synge has shown [17] that this limiting metric is one of constant curvature, namely the anti-de Sitter metric

$$g_{\mathrm{ads}} = (1 - \Lambda r^2/3)dt^2 - (1 - \Lambda r^2/3)^{-1}dr^2 - r^2 d\Omega^2. \tag{4.9}$$

Comparison of this with (3.1) and (3.2) shows that for this case ($\Lambda < 0$) it is necessary that

$$C/r^2 \to 1 \quad \text{as} \quad r \to \infty. \tag{4.10}$$

A suitable C for this case is thus

$$C_{\mathrm{inf}} \equiv (r + b)^2, \tag{4.11}$$

with b given by (4.4). However, even knowing that $g \to g_{\mathrm{ads}}$ does not suffice to determine α, because the presence of the $\Lambda(r+b)^2$ term in (3.1) makes the space-time more-and-more non-Newtonian as $r \to \infty$, and thus the Kepler-orbit requirement for distant test particles cannot be invoked to identify α with $2m$.

Incidentally, note that the strict monotonicity required of C by virtue of (3.3), together with the limiting values of C, shows that (2.2) is a diffeomorphism, so that the space-times whose A, B are given by (3.1) and (3.2) are isometric to Weyl's. Thus the only difference between Weyl's TST and those found here lies in the difference between their boundaries (see Section 7.2, below).

5. Singularities of the total space-times

In Ref. [9] the Kretschmann invariant $f \equiv R_{ijkm}R^{ijkm}$ is calculated for Weyl's metric. Transforming this via (2.2) and letting $r \to 0$ shows that f approaches

$$f_0 \equiv 12\alpha^2/b^2 + 24\Lambda^2/9, \tag{5.1}$$

so that for the values of b found to be required in the previous Section, f is bounded as $r \to 0$. Since all other scalar differential invariants are functions of f, it follows that there are no curvature singularities at the location of the point-mass.

However, (2.1) shows that the proper circumference of the circle $r = \varepsilon$ tends to $2\pi b > 0$ as $\varepsilon \downarrow 0$, while the proper radius of that circle is easily seen [from (2.1) and (3.2)] to tend to zero in that limit. These results are coordinate-independent: once a metric has been brought into the form of (2.1), the only transformations which preserve the form of (2.1) are $t = K\bar{t}+q$

$(K \neq 0)$, $r = h(\bar{r})$ $(h, h^{-1} \in C^\omega)$, neither of which alters the proper radius or proper circumference of $r = \varepsilon$.

Since the boundary at $r = 0$ is necessarily a point in each spatial section (see Section 1), these properties of the radius and circumference of $r = \varepsilon$ constitute a violation of elementary flatness at $r = 0$, and a fortiori a quasiregular singularity [18] which renders these TSTs inextendible.

6. Gravitational collapse

As pointed out in Ref. [14], the metric representing the exterior of a spherically symmetric star undergoing gravitational collapse to a point is subject to the same requirements as that of a point-mass except for those relating to its behavior at $r = 0$. It thus has the same form as that of a point-mass [i.e., (2.1) cum (3.1) and (3.2)], but different parameter values since C need only make A positive for all $r > r_b$, where r_b denotes the radial coordinate of the star's boundary. As $r_b \to 0$, this assures the positivity of A for all $r > 0$, so that no horizon, and a fortiori no black hole, can be formed at any stage of the collapse. Thus, just as in the case where $\Lambda = 0$, elimination of the invalid assumption regarding the location of the point-mass also deprives the Lake-Roeder black hole of the only mechanism suggested for its production.

7. Summary and discussion

7.1. The only physically sensible metrics for the point-mass when $\Lambda \neq 0$ are given by

$$g = Adt^2 - Bdr^2 - Cd\Omega^2,$$
$$A = 1 - \alpha/\sqrt{C} - \Lambda C/3, \quad B = C'^2/(4AC).$$

The form of C depends on Λ:

7.1.1. $\Lambda < 0$.

$C = (r + b)^2$ with b given by (4.4). Any positive, analytic, strictly monotonic increasing function of r having the same value of b and tending to infinity like r^2 can be used in place of this C and, with the space-time's boundary taken to be a line through the source, will give rise to an equivalent TST. The value of α must be determined from other considerations.

7.1.2. $0 < \Lambda < \Lambda_0$.

C is any analytic, strictly monotonic increasing function of r satisfying $C(0+) = b^2$ and $C(\infty) = k_{\max}^2$, where b is given by (4.5) and $k_{\max} = \sqrt{C_3}$. With the space-time's boundary taken to be a line through the source, all such C give rise to equivalent TSTs. The value of α must be determined from other considerations.

7.2. As noted in Section 1, the $r = 0$ boundary of the point-mass TSTs associated with the above metrics must necessarily be a point in each spatial section. Under the transformation $r^* = [C(r)]^{1/2}$ this boundary becomes $r^* = [C(0+)]^{1/2} = b$. That is to say, the locus $r^* = b$ is a point in each spatial section. But Weyl's assumption that the origin of the x, y, z coordinates is located at $r^* = 0$ automatically makes $r^* = b$ a two-sphere in each spatial section of his space-time. Thus, Weyl's assumption led to the attachment of a two-sphere, rather than a point, as the boundary of each spatial section of his space-time at $r^* = b$ - i.e., at the actual location of the point-mass. Consequently, his TST has a different singularity structure than that associated with the space-times having the above metrics. Since the metrics' derivation shows that the TSTs obtained here are the only possible ones for the point-mass when $\Lambda \neq 0$, it follows that his TST does not represent such an object. A fortiori, the same is true of the black hole-containing analytic extension of Weyl's TST found in 1977 by Lake and Roeder [9] and, independently, by Laue and Weiss [10] (to say nothing of the fact that this extension alters the topology of the boundary of Weyl's TST, and thus represents a different universe than the latter). This in turn means that this extension cannot serve as the limiting space-time of a spherically symmetric star undergoing collapse to a point. Thus, it is impossible to produce this extension by gravitational collapse. Lacking both a theoretical basis (i.e. a valid derivation from a set of postulates characterizing a specific universe) and a mechanism for its production, it follows that this extension, and with it its black hole, are merely artifacts of Weyl's error.

7.3. The relation between α and m is still open.

8. Acknowledgments

It is a pleasure to acknowledge helpful correspondence with G.F.R. Ellis and R. Geroch, as well as numerous conversations with B. O'Neill and R. Greene. Such errors as remain are mine alone.

Appendix A. Constraints on C required to make A positive

It is expedient to consider the following three intervals of Λ: $\Lambda < 0$; $0 < \Lambda < \Lambda_0$; $\Lambda_0 \leq \Lambda$, where $\Lambda_0 = 4/9\alpha^2$.

A.1. $\Lambda < 0$**.** It is shown in Ref. [9] (with r, $2m$ there replaced by $\sqrt{C}$, α respectively) that A is positive iff $C > C_0$, and vanishes only at $C = C_0$, where:

$$C_0 \equiv (\mu + \sqrt{\mu^2 - \Lambda^{-3}})^{1/3} + (\mu - \sqrt{\mu^2 - \Lambda^{-3}})^{1/3} > 0, \tag{A.1}$$

in which

$$\mu \equiv 3\alpha/(-2\Lambda) > 0. \tag{A.2}$$

Together with (3.3), the foregoing shows that in order to make $A > 0$, it is necessary and sufficient that

$$\sqrt{C(0+)} \equiv b \geq \sqrt{C_0} > 0. \tag{A.3}$$

A.2. $0 < \Lambda < \Lambda_0$. In this case, it is shown in Ref. [9] that A is positive iff:

$$C_2 < C < C_3, \tag{A.4}$$

and vanishes only when $C = C_2$ or when $C = C_3$, where:

$$\sqrt{C_2} \equiv (2/\sqrt{\Lambda}) \cos(\xi/3) > 0, \tag{A.5}$$

$$\sqrt{C_3} \equiv (2/\sqrt{\Lambda}) \cos(\xi/3 + 4\pi/3), \tag{A.6}$$

$$\cos(\xi) \equiv -3\alpha\sqrt{\Lambda}/2, \;\; \pi < \xi < 3\pi/2, \tag{A.7}$$

In combination with (3.3), the left-hand-most inequality in (A.4) requires that

$$b \geq \sqrt{C_2} > 0. \tag{A.8}$$

Appendix B. Extendibility of total space-times

Consider first the case $\Lambda < 0$ with k finite, so that the metric (g_{fin}) is given by (2.1), (3.1), (3.2) with $C(r) \equiv C_{\text{fin}}(r)$, where $C_{\text{fin}}(\infty) = k^2$. Consider also the same metric, but with $C(r) \equiv C_{\text{inf}}(r)$, where $C_{\text{inf}}(\infty) = \infty$, and let this second metric be designated by g_{inf}.

Thanks to the strict monotonicity of both C's, the mapping $\zeta : M_0 \to \zeta(M_0)$ by $C_{\text{fin}}(r) = C_{\text{inf}}(\bar{r})$ is an analytic diffeomorphism. It is also an isometry to the open proper submanifold $\bar{r} < C_{\text{fin}}^{-1}(k^2)$ of (M_0, g_{inf}). Finally, the boundary that is attached to (M_0, g_{fin}) at $r = \infty$ must be a two-sphere in each spatial section, rather than a point, since the latter would give rise to a second point-like singularity, which is inconsistent with the single point-mass hypothesis. Thus the space-time (M_0, g_{fin}) is isometric to a proper open submanifold of (M_0, g_{inf}), and its boundary is homeomorphic to its image. This means that the TST of (M_0, g_{fin}) is extendible to that of (M_0, g_{inf}).

Consider next the case $0 < \Lambda < \Lambda_0$. In this case the metric (g_k) is given by (2.1), (3.1), (3.2) with $C(r) \equiv C_k(r)$, where $C_k(0+) = C_2$ and $C_k(\infty) = k^2 < k_{\text{max}}^2 \equiv C_3$. Consider also the same metric, but with $C \equiv C_{\text{max}}(r)$, where $C_{\text{max}}(\infty) = k_{\text{max}}^2$, and let this second metric be denoted by g_{max}. Replacing fin by k and inf by max in the above proof of the extendibility of the TST of (M_0, g_{fin}), the same proof is seen to hold for this case as well, so that the TST of (M_0, g_k) is extendible to that of (M_0, g_{max}).

References

1. Tipler, F.J., Clarke, C.J.S., and Ellis, G.F.R., in *General Relativity and Gravitation: one hundred years after the birth of Albert Einstein*, A. Held, ed. (Plenum Press, 1980), p. 97.
2. Schmidt, B.G., Gen Rel. Gravitation **1** (1969) 269.
3. Geroch, R.P., J. Math Phys. **9** (1968) 450.
4. Dodson, C.T.J., Gen Rel. Gravitation **10** (1979) 969.
5. Hawking, S.W., and Ellis, G.F.R., *The large scale structure of space-time*, (Camb. Univ. Press, Cambridge, 1973), p. 276.
6. Weyl, H., *Raum, Zeit, Materie* (Singer, Berlin, 1918), pp. 224-226. This derivation is based on a form for the metric which Weyl had derived earlier [Ann. der Physik **54** (1917) 117, on p. 130] for the $\Lambda = 0$ case, viz.: $fdt^2 - \mu(dx_1^2 + dx_2^2 + dx_3^2) - L(x_1dx_1 + x_2dx_2 + x_3dx_3)$, with μ and L functions of r alone. There, he asserted that one could always choose the "scale" with which r is measured in such a way as to make $\mu \equiv 1$. This reduces the spatial metric to $dr^2 + r^2d\Omega^2 + Lr^2dr^2 \equiv (1 + Lr^2)dr^2 + r^2d\Omega^2$, which in fact can only be obtained from the general one (2.1) by a transformation equivalent to $r^* = [C(r)]^{1/2}$. Despite Weyl's use of this transformation for the case $\Lambda = 0$, which makes it impossible to determine the location of the point-mass in terms of r^*, he nonetheless asserted (on p. 131) that the point-mass in that case was located at $r^* = 2m$! Of course, he was familiar with Schwarzschild's and Flamm's papers, which likewise put the point-mass there, but it is curious that he did not assert this in connection with his derivation of the point mass for $\Lambda \neq 0$ some 90 pages later. [Weyl dropped the assertion that the point mass for $\Lambda = 0$ was at $r^* = 2m$ in subsequent editions of his book. It would seem to be worthwhile to look into Weyl's correspondence with Einstein, Flamm, Hilbert, etc., in the hopes of finding out what caused him to switch the mass-point position from $r^* = 2m$ to $r^* = 0$.]
7. Stavroulakis, N., Ann. Fond. Louis de Broglie **6** (1981) 287.
8. Sachs, R.K., and Wu, H., *General Relativity for Mathematicians* (Springer, New York, 1977), p. 27. The isometry involved must be orientation- and time-orientation-preserving. Here and henceforth, the unqualified words "isometry" and "isometric" will always be understood to refer to such isometries only.
9. Lake, K., and Roeder, R.C., Phys. Rev. D **15** (1977) 3513.
10. Laue, H., and Weiss, M., Phys. Rev. D **16** (1977) 3376.
11. By "extendibility" here will always be meant analytic extendibility.
12. Eiesland, J., Trans. Am. Math. Soc. **27** (1925) 213.
13. Weyl's result in Ref. [6] had $\Lambda r^2/6$ because of an error. See C. Lanczos, Phys. Zeitschr. XXIII (1922) 539.
14. Abrams, L.S., Can. J. Phys. **67** (1989) 919.
15. Ref. [6], p. 226: "Sie entspricht dem Fall, dass um das Zentrum ein Massenkugel liegt." ("This corresponds to the case, in which a spherical mass lies at the center.").
16. Doughty, N., Am. J. Phys. **49** (1981) 412.
17. Synge, J.L., *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), pp. 256, 281.
18. Ellis, G.F.R., and Schmidt, B., Gen. Rel. Gravitation **8** (1977) 915.

Minimax, 24345 Crestlawn Street, Woodland Hills, CA 91367, USA
E-mail address: `ag272@lafn.org`